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THE METHOD OF AVERAGING AND DOMAINS OF STABILITY FOR INTEGRAL MANIFOLDS*

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Abstract. Liapunov's direct method is a standard and effective approach to computing the domain of stability (or region of attraction) of an autonomous ordinary differential equation. In this paper the author investigates domains of stability of integral manifolds of solutions generated by nonlinear mechanical and electrical oscillatory systems with many degrees of freedom. These manifolds are families of solutions that exhibit stronger stability properties than individual solutions. The problem of estimating the domain of stability of an asymptotically stable integral manifold is reduced to computing the domain of stability of an associated autonomous system of differential equations. This is done by applying the method of averaging to the system generating the integral manifold thus removing angular and time dependences. The stability region of this associated system is then computed and a result is established showing that this region is contained in the stablity region of the original system. Several examples, including a coupled van der Pol system of oscillators, are considered.

1. Introduction. An integral manifold of solutions for an n-dimensional system of differential equations is represented geometrically as a hypersurface in $(n + 1)$-dimensional space with the property that if some value of a solution of the differential equation lies on the hypersurface then the entire solution will also lie on the hypersurface. The study of integral manifolds of solutions arises in the analysis of nonlinear oscillatory systems with m degrees of freedom of the form

$$\ddot{x}_i + \omega_i^2 x_i = \varepsilon X_i(t, x_1, \cdots, x_m, \dot{x}_1, \cdots, \dot{x}_m), \tag{1.1}$$

where $\varepsilon > 0$, and $\omega_i > 0$, for $i = 1, \cdots, m$. Local asymptotic stability of integral manifolds has been studied by a number of authors, for example, Bogoliubov and Mitropolsky [2, pp. 428–534], Hale [5, pp. 113–169], Hale and Stokes [7]. In this paper the author considers the question of nonlocal stability of integral manifolds and approaches the problem by Liapunov's direct method of estimating domains of stability in state space.

LaSalle and Lefschetz [10, pp. 56–71] have shown that Liapunov functions can be used effectively to compute the domains of stability of equilibrium points for autonomous differential equations. Zubov [15, pp. 196–223] has shown how to directly construct Liapunov functions in the neighborhood of asymptotically stable periodic solutions. The stability properties of an integral manifold for some n-dimensional systems can be studied, however, by introducing an appropriate polar-type change of coordinates to isolate the angular and radial behavior of the system. Under certain conditions the angular behavior of the system can then be "averaged" out and the system decoupled. One of the resulting lower order systems is autonomous. Bogoliubov and Mitropolsky [2, p. 501] and Hale [6] have shown that the stability of integral manifolds for the original system can be related to the stability properties of equilibrium points of the autonomous subsystem. The author shows here, furthermore, that the domains of stability of these

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equilibrium points are related to, and can be used to estimate, the domains of stability of integral manifolds for the unaveraged system. These results extend to the case of integral manifolds some results due to Loud and Sethna [11], Sethna [12], and Sethna and Moran [13].

Let E^p be a p-dimensional Euclidean space with vector norm $|x|$ such that $|x| = (x_1^2 + x_2^2 + \cdots + x_p^2)^{1/2}$; U^p a domain in E^p; and $S_p(x_0, \rho)$ the set of $x \in E^p$ such that $|x - x_0| \leqq \rho$. If $S \subset E^p$, let $S_p(S; \rho)$ be the set of all $x \in E^p$ such that for some $x_0 \in S$, $|x - x_0| \leqq \rho$.

Let $\Sigma = E \times E^k \times U^m \times V^n \times [0, \varepsilon_0]$, where U^m, V^n are domains in E^m, E^n, respectively, $\varepsilon_0 > 0$ is fixed, and let $C_0(\Sigma)$ be the set of continuous vector-valued functions $f(t, \theta, x, y, \varepsilon)$ on Σ. Two subclasses of functions will also be introduced. Let $A_{t,\varepsilon}(\theta, x, y)$ be the class of functions in $C_0(\Sigma)$ that, for each fixed $\varepsilon \in [0, \varepsilon_0]$, are almost periodic in t, uniformly for $(\theta, x, y) \in E^k \times U^m \times V^n$. For the properties of almost periodic functions see Bohr [3]. Let $P^{\omega}_{\theta,\varepsilon}(t, x, y)$ be the class of all functions in $C_0(\Sigma)$ that are multiply periodic in θ, of vector period ω, independent of t, x, y, where $\omega = (\omega_1, \omega_2, \cdots, \omega_k)$, $\omega_i > 0$ for $i = 1, \cdots, k$.

Let $B_0(D)$ be the set of functions in $C_0(\Sigma)$ such that $|f(t, \theta, x, y, \varepsilon)| \leqq D$. Set $\lambda(\varepsilon)$ to be the Lipschitz constant for $f \in C_0(\Sigma)$ with respect to θ, x, y, where

$$|f(t, \theta, x, y, \varepsilon) - f(t, \theta', x', y', \varepsilon)| \leqq \lambda(\varepsilon)\{|\theta - \theta'| + |x - x'| + |y - y'|\}.$$

$\lambda(\varepsilon)$ is assumed to be continuous and bounded in ε, uniformly in t. In particular, suppose $\lambda(\varepsilon) \leqq \lambda$ for all $\varepsilon \in [0, \varepsilon_0]$. Finally, let $\mathrm{Lip}(\theta, x, y; \lambda(\varepsilon))$ be the set of all $f \in C_0(\Sigma)$ that are Lipschitzian in θ, x, y with Lipschitz constant $\lambda(\varepsilon)$, for each $\varepsilon \in [0, \varepsilon_0]$.

If A is a matrix applied to $x \in E^p$, then let

$$\|A\| = \left(\sum_{i,j} |a_{ij}|^2\right)^{1/2}.$$

Note that

$$\|A\| \leqq \sum_{i,j} |a_{ij}|.$$

Finally, if S is any set then the boundary of S will be denoted by ∂S.

2. Preliminary results. Consider a system of differential equations of the type

$$\frac{dx}{dt} = X(t, x, \varepsilon). \tag{2.1}$$

Here x, $X \in E^n$, an n-dimensional Euclidean space, $t \in E$, and $0 \leqq \varepsilon \leqq \varepsilon_0$, for some fixed ε_0. X is assumed to be continuous in the variables t, x, ε. Let the solution of (1.1), passing through the point (x_0, t_0), be given by $x(t, \varepsilon)$, where $x(t_0, \varepsilon) = x_0$.

Let $\varepsilon > 0$ be fixed and suppose that there exists an $(s + 1)$-dimensional surface $M(\varepsilon)$, $s \leqq n$, in the (x, t)-space given in parametric form by

$$M(\varepsilon) = \{(x, t) | x = F(t, c_1, \cdots, c_s, \varepsilon), t \in (-\infty, \infty)\}, \tag{2.2}$$

where F is continuous in $t, c_i, \varepsilon, i = 1, \cdots, s$. The surface $M(\varepsilon)$ is called an $(s + 1)$-dimensional *integral manifold* for (2.1) if any solution $x(t, \varepsilon)$ through $(x_0, t_0) \in M(\varepsilon)$ has the property that $(x(t, \varepsilon), t) \in M(\varepsilon)$ for $t \in (-\infty, \infty)$.

An integral manifold for (2.1) is called *stable* if for any $\eta > 0$ there exists a $\delta > 0$ such that if $d((x_0, t_0), M(\varepsilon)) < \delta$ then, for $t \geqq t_0$, $d((x(t, \varepsilon), t), M(\varepsilon)) < \eta$, where $x(t_0, \varepsilon) = x_0$ and $d((x, t), M(\varepsilon))$ represents the distance between the point (x, t) and the set $M(\varepsilon)$ in $(n + 1)$-space. $M(\varepsilon)$ is *asymptotically* stable if $M(\varepsilon)$ is stable and

$$\lim_{t \to +\infty} d((x(t, \varepsilon), t), M(\varepsilon)) = 0. \tag{2.3}$$

The *domain of stability*, Δ_0, for an asymptotically stable integral manifold $M(\varepsilon)$ is the set of all $x_0 \in E^n$ such that if $x(t, \varepsilon)$ is a solution of (2.1) with $x(0, \varepsilon) = x_0$, then (2.3) holds.

A simple example of an integral manifold can be given by considering the autonomous system

$$\dot{x} = X(x). \tag{2.4}$$

Suppose that $p(t)$ is a periodic solution of (2.4) of period T. For any constant c, $p(t + c)$ is also a periodic solution of (2.4) with period T. Let $x = F(t, c) = p(t + c)$. Then,

$$M = \{(x, t) | x = F(t, c), t \in (-\infty, \infty)\}$$

is an integral manifold for (2.4) and forms a cylinder in (x, t)-space.

By increasing the dimension of the system, equations of the form (1.1) are of the general class (2.1). Krylov and Bogoliubov [9, p. 87] and Bogoliubov and Mitropolsky [2, p. 445] have studied systems of the form (1.1) by introducing either rectangular or polar type coordinates. Depending on the transformation used the coupled system of oscillators could be reduced to a general system of the type (2.1), and in fact took either of the general forms

$$\dot{x} = \varepsilon X(t, x), \tag{2.5}$$

where x, X are $2m$-vectors, or

$$\begin{aligned} \dot{\theta} &= d + \varepsilon \Theta(t, \theta, \rho), \\ \dot{\rho} &= \varepsilon R(t, \theta, \rho), \end{aligned} \tag{2.6}$$

where $d = (1, 1, \cdots, 1)$ and θ, ρ are m-vectors.

In this paper a more general system covering systems of the form (2.5) and (2.6) will be studied. In particular, consider the system

$$\begin{aligned} \frac{d\theta}{dt} &= d(\varepsilon) + \varepsilon \Theta(t, \theta, x, y, \varepsilon), \\ \frac{dx}{dt} &= \varepsilon X(t, \theta, x, y, \varepsilon), \\ \frac{dy}{dt} &= Ay + \varepsilon Y(t, \theta, x, y, \varepsilon), \end{aligned} \tag{2.7}$$

with the following assumptions:

H1. $0 \leqq \varepsilon \leqq \varepsilon_0$; $\Theta, d(\varepsilon), \theta \in E^k$; $x, X \in U^m \subset E^m$ and $y, Y \in V^n \subset E^n$; $d(\varepsilon) = 1 + O(\varepsilon)$ where $1 \equiv (1, 1, \cdots, 1) \in E^k$.

H2. Θ, X, Y, and their first and second partial derivatives with respect to θ, x, y are bounded and uniformly continuous with respect to t, θ, x, y, ε, i.e., Θ, X, $Y \in B_0(M)$ for some M, as well as the first and second partials.

H3. $\Theta, X, Y \in A_{t,\varepsilon}(\theta, x, y) \cap P^{\omega}_{\theta,\varepsilon}(t, x, y)$.

Remark. Assumption H2 implies that Θ, X, $Y, \in \mathrm{Lip}\,(\theta, x, y; \lambda(\varepsilon))$ for some function of $\lambda(\varepsilon)$. This assumption can be weakened, but for the purposes of this paper H2 is sufficient. For further discussion the reader is referred to Hale [6] for results with weaker assumptions.

DEFINITION 2.1. Let $f(t, \theta, x, y, \varepsilon) \in C_0(\Sigma)$. Define the *average*, f_0, of f with respect to t and θ by the relation

$$f_0(x, y) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t + s, \theta + s, x, y, 0)\, ds, \tag{2.8}$$

provided the limit exists uniformly with respect to $(t, \theta, x, y) \in E \times E^k \times U^m \times V^n$ and is independent of t and θ. In (2.8), $\theta + s = (\theta_1 + s, \theta_2 + s, \cdots, \theta_k + s)$. A sufficient condition for the limit to exist independently of t and θ can be given if f is also periodic in t with period ω_0, say. This result is given in Hale [5, p. 119]. In particular, let $\omega_0, \omega_1, \cdots, \omega_k$ be the periods of f with respect to t, θ_i, $i = 1, \cdots, k$. Suppose that $\omega_0^{-1}, \omega_1^{-1}, \cdots, \omega_k^{-1}$ are linearly independent over the integers. Expand f in a Fourier series (with equality in the L_2 sense).

$$\begin{aligned} f(t, \theta, x, y, \varepsilon) &= c_{0\cdots 0}(x, y, \varepsilon) \\ &\quad + \sum_{I_k} c_{n_0\cdots n_k}(x, y, \varepsilon) \exp\left[2\pi i\left(\frac{n_0}{\omega_0}t + \cdots + \frac{n_k}{\omega_k}\theta_k\right)\right], \end{aligned}$$

where $I_k = \{(n_0, \cdots, n_k) | n_i, i = 0, 1, \cdots, k, \text{ integers}, n_0^2 + \cdots + n_k^2 \neq 0\}$. Averaging over $[0, T]$,

$$\begin{aligned} \frac{1}{T}\int_0^T f(t + s, \theta + s, x, y, 0)\, ds = c_{0\cdots 0}(x, y, 0) &+ \sum_{I_k} c_{n_0\cdots n_k}(x, y, 0) \\ &\cdot \exp\left[2\pi i\left(\frac{n_0}{\omega_0}t + \cdots + \frac{n_k}{\omega_k}\theta_k\right)\right] \\ &\cdot \left[\frac{1}{T}\int_0^T \exp 2\pi i\left(\frac{n_0}{\omega_0} + \cdots + \frac{n_k}{\omega_k}\right) s\, ds\right]. \end{aligned}$$

Since $\omega_0^{-1}, \omega_1^{-1}, \cdots, \omega_k^{-1}$ are linearly independent over the integers $n_0/\omega_0 + \cdots + n_k/\omega_k \neq 0$, the right-hand side becomes in the limit $c_{0\cdots 0}(x, y, 0)$, which is independent of t and θ.

One further assumption will now be made for (2.7).

H4. Suppose there exists a point $x_0 \in U^m$ such that $X_0(x_0, 0) = 0$. Let $(\partial X_0/\partial x)(x_0, 0)$ and A have eigenvalues with negative real parts.

System (2.7) now satisfies sufficient conditions for existence and asymptotic stability of an integral manifold. In fact Hale [6] proves the following.

THEOREM 2.2. *Under the assumptions* H1–H4 *for system* (2.7) *there exist positive constants* ε_1, σ_0, c, γ *and functions* $f(t,\theta,\varepsilon)$, $g(t,\theta,\varepsilon)$ *of dimensions* m *and* n, *respectively, satisfying the following properties*:

P1: *For* $0 \leqq \varepsilon \leqq \varepsilon_1$, $f(t,\theta,\varepsilon)$, $g(t,\theta,\varepsilon)$ *are continuous in* $(t,\theta,\varepsilon) \in E \times E^k \times [0,\varepsilon_1]$, *and are multiply periodic in* θ *with vector period* ω. *For each fixed* $\varepsilon \in [0,\varepsilon_1]$, f *and* g *are almost periodic in* t *uniformly with respect to* $\theta \in E^k$. $f(t,\theta,0) = x_0$, $g(t,\theta,0) = 0$ *and* $|f(t,\theta,\varepsilon) - x_0| < \sigma_0$, $|g(t,\theta,\varepsilon)| < \sigma_0$ *for all* t, θ *and* $\varepsilon \in (0,\varepsilon_1]$.

P2: *For each fixed* $\varepsilon \in [0,\varepsilon_1]$, $x = f(t,\theta,\varepsilon)$, $y = g(t,\theta,\varepsilon)$ *is a parametric representation of an integral manifold for* (2.7). *Call this integral manifold* $M(\varepsilon)$.

P3: *Let* $(x_1,y_1) \in S((x_0,0);\sigma_0)$ *and* $\theta_1 \in E^k$ *be arbitrary. Suppose that* $(\theta(t),x(t),y(t))$ *is the solution of* (2.7) *such that* $\theta(t_1) = \theta_1$, $x(t_1) = x_1$, $y(t_1) = y_1$ *for some time* t_1. *Then for* $t \geqq t_1$,

$$|x(t) - f(t,\theta(t),\varepsilon)| \leqq c\{\exp[-\varepsilon\gamma(t - t_1)]\}\{|x_1 - f(t_1,\theta_1,\varepsilon)| + |y_1 - g(t_1,\theta_1,\varepsilon)|\},$$

$$|y(t) - g(t,\theta(t),\varepsilon)| \leqq c\{\exp[-\gamma(t - t_1)]\}\{|x_1 - f(t_1,\theta_1,\varepsilon)| + |y_1 - g(t_1,\theta_1,\varepsilon)|\}.$$

The definition of domain of stability must now be specialized somewhat for system (2.7). The following definition will be used throughout this paper.

DEFINITION 2.3. The *domain of stability* of the *integral* manifold $M(\varepsilon)$ of (2.7) is defined to be

$$\Delta_0 = \{(\theta_0,x_0,y_0)|\theta_0 \in E^k \text{ arbitrary, and if } (\theta(t),x(t),y(t)) \text{ is the solution of (2.7) such that } \theta(0) = \theta_0,\ x(0) = x_0,\ y(0) = y_0, \text{ then } |x(t) - f(t,\theta(t),\varepsilon)| \to 0 \text{ and } |y(t) - g(t,\theta(t),\varepsilon)| \to 0 \text{ as } t \to \infty\}.$$

Remark. From Hale's result (Theorem 2.2) it is clear that Δ_0 is nonempty since for $t_1 = 0$,

$$E^k \times S((x_0,0);\sigma_0) \subset \Delta_0.$$

In general Δ_0 encompasses a much larger region. This then is the main problem: Estimate the size of Δ_0, if possible.

There is another result proved by Hale [5, p. 116] that is closely related to Theorem 2.2. This result lays the groundwork for the rest of the paper.

THEOREM 2.4. *Let* (2.7) *satisfy the assumptions* H1–H4. *Let* $\Theta^*(t,\theta,x,y,\varepsilon)$, $X^*(t,\theta,x,y,\varepsilon)$ *satisfy the same conditions as* Θ, X *in Theorem* 2.2. *It is further assumed that*

$$\Theta_0^*(x,y,\varepsilon) = 0, \qquad X_0^*(x,y,\varepsilon) = 0.$$

Then the conclusion of Theorem 2.2 *also holds for*

$$\begin{aligned}\dot\theta &= d(\varepsilon) + \varepsilon\Theta(t,\theta,x,y,\varepsilon) + \varepsilon\Theta^*(t,\theta,x,y,\varepsilon),\\ \dot x &= \varepsilon X(t,\theta,x,y,\varepsilon) + \varepsilon X^*(t,\theta,x,y,\varepsilon),\\ \dot y &= Ay + \varepsilon Y(t,\theta,x,y,\varepsilon).\end{aligned} \tag{2.9}$$

Returning to systems (2.5) and (2.6), suppose that the following averages exist, independent of t and θ:

$$
\begin{aligned}
X_0(x) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T X(t,x)\,dt, \\
\Theta_0(\rho) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T \Theta(t+s,\theta+s,\rho)\,ds, \\
R_0(\rho) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T R(t+s,\theta+s,\rho)\,ds.
\end{aligned} \tag{2.10}
$$

One now has the averaged systems

$$\dot{x} = \varepsilon X_0(x) \tag{2.11}$$

and

$$\dot{\theta} = d + \varepsilon\Theta_0(\rho), \qquad \dot{\rho} = \varepsilon R_0(\rho). \tag{2.12}$$

Considering for the moment system (2.6) one can rewrite (2.6) as

$$
\begin{aligned}
\dot{\theta} &= d + \varepsilon\Theta_0(\rho) + \varepsilon\Theta^*(t,\theta,\rho), \\
\dot{\rho} &= \varepsilon R_0(\rho) + \varepsilon R^*(t,\theta,\rho),
\end{aligned} \tag{2.13}
$$

where

$$
\begin{aligned}
\theta^*(t,\theta,\rho) &= \Theta(t,\theta,\rho) - \Theta_0(\rho), \\
R^*(t,\theta,\rho) &= R(t,\theta,\rho) - R_0(\rho),
\end{aligned} \tag{2.14}
$$

and

$$\Theta_0^*(\rho) = R_0^*(\rho) = 0, \qquad \text{from (2.10).}$$

Theorem 2.2 now applies to (2.12) and Theorem 2.4 to (2.13), without the third equation in the system. This suggests that one might compare in some manner the solutions of (2.12) against those of (2.6). The solutions of (2.12) yield a first order approximation in ε. These approximations have been studied extensively by Krylov and Bogoliubov [9, p. 8] and Bogoliubov and Mitropolsky [2, p. 387].

System (2.5) is a particular case of (2.7), but without the θ and y subsystems. Loud and Sethna [11], Sethna [12], and Sethna and Moran [13] have proved some results that relate the domains of stability of asymptotically stable equilibrium points of (2.11) to the domains of stability of periodic and almost periodic solutions of (2.5). It is these results the author wishes to extend to the case of integral manifolds of (2.7). A comparison theorem will be proved in the next section which will be the fundamental tool. The motivation of the theorem was the formulation of Theorem 2.4, and in particular system (2.9).

3. Fundamental comparison theorem. In this section a proof will be given of a comparison theorem between the solutions of system (2.7) and those of

$$
\begin{aligned}
\frac{d\theta}{dt} &= d(\varepsilon) + \varepsilon\Theta_0(x, y),\\
\frac{dx}{dt} &= \varepsilon X_0(x, y), \qquad (3.1)\\
\frac{dy}{dt} &= Ay + \varepsilon Y(t, \theta, x, y, \varepsilon).
\end{aligned}
$$

System (3.1) will be referred to as the *averaged system*. This result will extend a theorem on averaging due to Bogoliubov and Mitropolsky [2, p. 429] to systems of the form (2.7). In order to simplify the proof somewhat several lemmas are needed.

LEMMA 3.1. *Let $P(t)$ be continuous for $t \geqq 0$ and satisfy $|P(t)| \leqq M$ and let the eigenvalues of the matrix A all have negative real parts. Then there exist positive constants, b, c such that*

$$\|e^{At}\| \leqq c\, e^{-bt}$$

for $t \geqq 0$, and the solution of

$$\frac{dx}{dt} = Ax + \varepsilon P(t)$$

satisfies

$$|X(t)| \leqq c|x_0|\, e^{-bt} + \frac{\varepsilon M c}{b},$$

where $x_0 = x(0)$.

The proof of this lemma is omitted, but it follows as a direct consequence of the variation of parameters formula.

LEMMA 3.2. *Let $f(t)$ be a continuous function and $\alpha_r > 0$ for $r = 1, 2, \cdots, n$. Set*

$$U_n = \{(t_1, \cdots, t_n) | t_1 + t_2 + \cdots + t_n \leqq 1,\ t_i > 0,\ i = 1, \cdots, n\}.$$

Then Dirichlet's integral is given by

$$\int \cdots \int_{U_n} f(t_1 + t_2 + \cdots + t_n) t_1^{\alpha_1 - 1} t_2^{\alpha_2 - 1} \cdots t_n^{\alpha_n - 1}\, dt_1\, dt_2 \cdots dt_n = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_n)}{\Gamma(\alpha_1 + \alpha_2 + \cdots + \alpha_n)} \int_0^1 f(s) s^\beta\, ds,$$

where $\beta = (\sum_{i=1}^n \alpha_i) - 1$ and $\Gamma(\alpha)$ is the gamma function.

Proof. See Whittaker and Watson [14, p. 258].

LEMMA 3.3. *Let $x \in E^n$ and suppose that*

$$I_a^n = \int_{E^n} \left| \frac{\partial \Delta_a(x)}{\partial x} \right| dx,$$

where

$$\Delta_a(x) = \begin{cases} A_a\{1 - |x|^2/a^2\}^2, & |x| \leqq a, \\ 0, & |x| > a, \end{cases}$$

and A_a is the normalizing constant such that

$$\int_{E^n} \Delta_a(x)\, dx = 1.$$

Then

$$A_a = \frac{\Gamma(n/2)n(n+2)(n+4)}{16 \cdot \pi^{n/2} a^n}$$

and

$$I_a^n \leqq \frac{3}{2} \frac{n^2(n+2)(n+4)}{\pi^{1/2} a(n+1)(n+3)} \frac{\Gamma(n/2)}{\Gamma((n+1)/2)}.$$

Proof. The first problem is to compute A_a. To do this note that

$$\int_{E_n} \Delta_a(x)\, dx = A_a \int \cdots \int_{|x| \leqq a} \left\{ 1 - \frac{2}{a^2} \sum_{i=1}^{n} x_i^2 + \frac{1}{a^4} \sum_{i=1}^{n} \sum_{j=1}^{n} x_i^2 x_j^2 \right\} dx_1 \cdots dx_n. \tag{3.2}$$

Use Lemma 3.1 to compute the individual integrals. To illustrate the technique consider

$$\int \cdots \int_{|x| \leqq a} dx_1\, dx_2 \cdots dx_n.$$

The set of x such that $|x| \leqq a$ is equivalent to the set of $x = (x_1, \cdots, x_n)$ such that

$$\left(\frac{x_1}{a}\right)^2 + \cdots + \left(\frac{x_n}{a}\right)^2 \leqq 1.$$

Then

$$\int \cdots \int_{|x| \leqq a} dx_1 \cdots dx_n = 2^n \int \cdots \int_{\substack{|x| \leqq a \\ 0 \leqq x_i \\ i = 1, \cdots, n}} dx_1 \cdots dx_n.$$

Set $t_i = (x_i/a)^2$. Then $x_i = a t_i^{1/2}$ and $dx_i = (a/2) t_i^{1/2 - 1}\, dt_i$. Thus

$$\begin{aligned} \int \cdots \int_{|x| \leqq a} dx_1 \cdots dx_n &= 2^n \int \cdots \int_{\substack{|x| \leqq a \\ 0 \leqq x_i \\ i = 1, \cdots, n}} dx_1 \cdots dx_n \\ &= a^n \int \cdots \int_{t_1 + \cdots + t_n \leqq 1} t_1^{1/2 - 1} \cdots t_n^{1/2 - 1}\, dt_1 \cdots dt_n \\ &= \frac{2a^n}{n} \frac{\pi^{n/2}}{\Gamma(n/2)} \end{aligned}$$

by Lemma 3.2 and the fact that $\Gamma(\frac{1}{2}) = \pi^{1/2}$. By similar arguments it is not hard to show that for each given i, j,

$$\int_{|x| \leq a} \cdots \int x_i^2 \, dx_1 \cdots dx_n = \frac{a^{n+2}}{n+2} \frac{\pi^{n/2}}{\Gamma((n+2)/2)}$$

and

$$\int_{|x| \leq a} \cdots \int x_i^2 x_j^2 \, dx_1 \cdots dx_n = \frac{a^{n+4}}{2(n+4)} \frac{\pi^{n/2}}{\Gamma((n+4)/2)}.$$

From these integrals one gets, by (3.2),

$$\int_{E^n} \Delta_a(x) \, dx = \frac{A_a a^n \pi^{n/2} 16}{\Gamma(n/2) n(n+2)(n+4)},$$

where repeated use has been made of the recurrence formula $\Gamma(1 + z) = z\Gamma(z)$, with $z = n/2$.

In order that

$$\int_{E^n} \Delta_a(x) \, dx = 1$$

choose

$$A_a = \frac{\Gamma(n/2) n(n+2)(n+4)}{16 \pi^{n/2} a^n}.$$

This proves the first part of the result.

Next, it is clear that

$$\int_{E^n} \left| \frac{\partial \Delta_a(x)}{\partial x} \right| dx \leq \sum_{j=1}^{n} \int_{E^n} \left| \frac{\partial \Delta_a(x)}{\partial x_j} \right| dx.$$

Then one only needs to estimate, for each j,

$$\begin{aligned} \int_{E^n} \left| \frac{\partial \Delta_a(x)}{\partial x_j} \right| dx &= \frac{4A_a}{a^2} \int_{|x| \leq a} \cdots \int \left\{ 1 - \frac{|x|^2}{a^2} \right\} |x_j| \, dx \\ &= \frac{4A_a}{a^2} \left[\int_{|x| \leq a} \cdots \int |x_j| \, dx \right] - \frac{4A_a}{a^4} \left[\sum_{i=1}^{n} \int_{|x| \leq a} \cdots \int x_i^2 |x_j| \, dx \right]. \end{aligned}$$

By previous arguments,

$$\begin{aligned} \int_{|x| \leq a} \cdots \int |x_j| \, dx &= 2^n \int_{\substack{|x| \leq a \\ 0 \leq x_k \\ k = 1, \cdots, n}} \cdots \int x_j \, dx_1 \cdots dx_n \\ &= \frac{2a^{n+1} \pi^{(n-1)/2}}{(n+1) \Gamma((n+1)/2)} \end{aligned}$$

and

$$\int_{|x|\leqq a} \cdots \int x_i^2 |x_j| \, dx_1 \cdots dx_n = \frac{2a^{n+3}\pi^{(n-1)/2}}{(n+1)(n+3)\Gamma((n+1)/2)}$$

Then

$$\int_{E^n} \left| \frac{\partial \Delta_a(x)}{\partial x_j} \right| dx = \frac{3}{2} \cdot \frac{1}{\pi^{1/2}} \cdot \frac{1}{a} \cdot \frac{n(n+2)(n+4)}{(n+1)(n+3)} \frac{\Gamma(n/2)}{\Gamma((n+1)/2)}.$$

Finally,

$$\int_{E^n} \left| \frac{\partial \Delta_a(x)}{\partial x} \right| dx \leqq \frac{3}{2} \frac{n^2(n+2)(n+4)}{\pi^{1/2} a(n+1)(n+3)} \frac{\Gamma(n/2)}{\Gamma((n+1)/2)},$$

and the result is complete.

Remark. For $n = 1, 2$ the estimates in Lemma 3.3 become, respectively,

$$\int_{-\infty}^{\infty} \left| \frac{d\Delta_a(x)}{dx} \right| dx \leqq \frac{45}{16a},$$

and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \frac{\partial \Delta_a(x)}{\partial x} \right| dx \leqq \frac{96}{5\pi a}.$$

The next lemma is an extension of a result due to Bogoliubov and Mitropolsky [2, p. 429].

LEMMA 3.4. *Let* $W(t, \theta, x, y, \varepsilon) \in C_0(\Sigma) \cap A_{t,\varepsilon}(\theta, x, y) \cap P^{\omega}_{\theta,\varepsilon}(t, x, y) \cap \mathrm{Lip}\,(\theta, x, y; \lambda(\varepsilon))$, *where* $\lambda(\varepsilon) \leqq \lambda$ *on* $(0, \varepsilon_0]$. *Finally suppose* $W_0(x, y) = 0$ *for* $(x, y) \in U^m \times V^n$. *Then, given* $L > 0, \mu > 0$ *and* D_0 *a subset of* $U^m \times V^n$ *such that, for some* $\alpha > 0$, $S_{m+n}(D_0; \alpha) \subset U^m \times V^n$, ($D_0$ *could also be chosen as a compact subset of* $U^m \times V^n$), *there exists a function* $u(t, \theta, x, y)$, *and a function* $G(\varepsilon)$ *such that* $G(\varepsilon) \to 0$ *as* $\varepsilon \to 0$ *and if* $0 \leqq t \leqq L/\varepsilon$ *then for* $\theta \in E^k$ *and* $(x, y) \in D_0$ *one has*

(i) $u(0, \theta, x, y) = 0$,

(ii) *the first partials of* u *with respect to* t, θ, x, y *exist and*

$$\left| \frac{\partial u}{\partial t}(t, \theta, x, y) + \sum_{p=1}^{k} \frac{\partial u}{\partial \theta_p}(t, \theta, x, y) - W(t, \theta, x, y, 0) \right| < \mu,$$

(iii) $$|\varepsilon u|, \left\| \varepsilon \frac{\partial u}{\partial \theta} \right\|, \left\| \varepsilon \frac{\partial u}{\partial x} \right\| \left\| \varepsilon \frac{\partial u}{\partial y} \right\| \leqq G(\varepsilon).$$

Proof. Choose a so that $a < \mu/(3\lambda)$ and construct the following functions:

$$\Gamma_a(\theta) = \begin{cases} d_1\{1 - |\theta|^2/a^2\}^2, & |\theta| \leqq a, \\ 0, & |\theta| > a, \end{cases}$$

$$\Delta_a(x) = \begin{cases} d_2\{1 - |x|^2/a^2\}^2, & |x| \leqq a, \\ 0, & |x| > a, \end{cases}$$

$$\delta_a(y) = \begin{cases} d_3\{1 - |y|^2/a^2\}^2, & |y| \leqq a, \\ 0, & |y| > a, \end{cases}$$

where

$$d_1 = \frac{\Gamma(k/2)k(k+2)(k+4)}{16\pi^{k/2}a^k},$$

$$d_2 = \frac{\Gamma(m/2)m(m+2)(m+4)}{16\pi^{m/2}a^m},$$

and

$$d_3 = \frac{\Gamma(n/2)n(n+2)(n+4)}{16\pi^{n/2}a^n}.$$

Then, from Lemma 3.3,

$$\int_{E^k} \left|\frac{\partial \Gamma_a(\theta)}{\partial \theta}\right| d\theta \leqq \frac{3}{2} \frac{k^2(k+2)(k+4)\Gamma(k/2)}{\pi^{1/2}a(k+1)(k+3)\Gamma((k+1)/2)},$$

$$\int_{E^m} \left|\frac{\partial \Delta_a(x)}{\partial x}\right| dx \leqq \frac{3}{2} \frac{m^2(m+2)(m+4)\Gamma(m/2)}{\pi^{1/2}a(m+1)(m+3)\Gamma((m+1)/2)}, \tag{3.3}$$

$$\int_{E^n} \left|\frac{\partial \delta_a(y)}{\partial y}\right| dy \leqq \frac{3}{2} \frac{n^2(n+2)(n+4)\Gamma(n/2)}{\pi^{1/2}a(n+1)(n+3)\Gamma((n+1)/2)}.$$

Define the following function of (t, θ, x, y):

$$u(t,\theta,x,y) = \int_{E^k \times D} \Gamma_a(\theta - \alpha)\Delta_a(x - \beta)\delta_a(y - \gamma) \cdot \left\{\int_0^t W(s, s + \alpha - t, \beta, \gamma, 0)\, ds\right\} d\alpha\, d\beta\, d\gamma, \tag{3.4}$$

where $D = U^m \times V^n$.

Since $W_0(x, y) = 0$ there exists a function $f(t)$ such that $f(t) \to 0$ as $t \to +\infty$, $f(0)$ finite, and

$$\left|\int_0^t W(s, r + s, x, y, 0)\, ds\right| \leqq tf(t) \tag{3.5}$$

for all $t \geqq 0$, uniformly in r, x, y. Then from (3.4) and (3.5), one has for $(t, \theta, x, y) \in E \times E^k \times D$ that

$$|u(t,\theta,x,y)| \leqq tf(t), \tag{3.6}$$

and conclusion (i) follows as an immediate consequence of (3.6), i.e.,

$$u(0,\theta,x,y) = 0 \tag{3.7}$$

since $f(0)$ is finite.

For $(t, \theta, x, y) \in E \times E^k \times D$, using (3.4), we have

$$\left\|\frac{\partial u}{\partial \theta}\right\| \leqq \sum_{j=1}^{k} \left|\frac{\partial u}{\partial \theta_j}\right| \leqq ktf(t)I_a^k \tag{3.8a}$$

since

$$\int_{E^k}\left|\frac{\partial\Gamma_a(\theta)}{\partial\theta_j}\right|d\theta \leqq \int_{E^k}\left|\frac{\partial\Gamma_a(\theta)}{\partial\theta}\right|d\theta$$

in (3.4) and I_a^k was defined in Lemma 3.3. Furthermore,

$$\left\|\frac{\partial u}{\partial x}\right\| \leqq mtf(t)I_a^m,$$

(3.8b)

$$\left\|\frac{\partial u}{\partial y}\right\| \leqq ntf(t)I_a^n.$$

Equation (3.4) can be rewritten as

$$\begin{aligned} u(t,\theta,x,y) = \int_{E^k\times D} & \Gamma_a(\phi)\Delta_a(x-\beta)\delta_a(y-\gamma) \\ & \cdot\left\{\int_0^t W(s,s+\theta-\phi-t,\beta,\gamma,0)\,ds\right\}\,d\phi\,d\beta\,d\gamma. \end{aligned} \tag{3.9}$$

Taking partials of (3.9) with respect to t gives

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,\theta,x,y) \\ &= \int_{E^k\times D}\Gamma_a(\phi)\Delta_a(x-\beta)\delta_a(y-\gamma)W(t,\theta-\phi,\beta,\gamma,\varepsilon)\,d\phi\,d\beta\,d\gamma \\ &- \sum_{p=1}^{k}\int_{E^k\times D}\Gamma_a(\phi)\Delta_a(x-\beta)\delta_a(y-\gamma) \\ &\cdot\left\{\int_0^t \frac{\partial W}{\partial\theta p}(s,s+\theta-\phi-t,\beta,\gamma,0)\,ds\right\}\,d\phi\,d\beta\,d\gamma. \end{aligned} \tag{3.10}$$

Taking partials of (3.9) with respect to each coordinate θ_p of θ gives

$$\begin{aligned} \frac{\partial u}{\partial\theta_p}(t,\theta,x,y) = \int_{E^k\times D} & \Gamma_a(\phi)\Delta_a(x-\beta)\delta_a(y-\gamma) \\ & \cdot\left\{\int_0^t \frac{\partial W}{\partial\theta_p}(s,s+\theta-\phi-t,\beta,\gamma,0)\,ds\right\}\,d\phi\,d\beta\,d\gamma. \end{aligned} \tag{3.11}$$

Let (x, y) be any point of D_0. Then there exists a number $a = a(D_0)$ such that $S_{m+n}((x, y); a) \subset D$ for all $(x, y) \in D_0$. For any $(x, y) \in D_0$ and an arbitrary θ,

$$\begin{aligned} &\int_{E^k\times D}\Gamma_a(\theta-\alpha)\Delta_a(x-\beta)\delta_a(y-\gamma)\,d\alpha\,d\beta\,d\gamma \\ &= \left[\int_{E^k}\Gamma_a(\theta-\alpha)\,d\alpha\right]\left[\int_D \Delta_a(x-\beta)\delta_a(y-\gamma)\,d\beta\,d\gamma\right] \\ &= \int_{\substack{|x-\beta|\leqq a\\ |y-\gamma|\leqq a}}\Delta_a(x-\beta)\delta_a(y-\gamma)\,d\beta\,d\gamma \end{aligned}$$

since $\int_{E^k} \Gamma_a(\theta - \alpha)\, d\alpha = \int_{E^k} \Gamma_a(\theta)\, d\theta = 1$ by the definition of Γ_a, d_1 and Lemma 3.3, as well as the fact that $\Delta_a(x - \beta) = \delta_a(y - \gamma) = 0$ for $|x - \beta| > a$ and $|y - \gamma| > a$. From the definition of Δ_a, δ_a and d_2, d_3, the normalizing constants,

$$\int_{\substack{|x-\beta| \leqq a \\ |y-\gamma| \leqq a}} \Delta_a(x - \beta)\delta_a(y - \gamma)\, d\beta\, d\gamma = \left[\int_{E^m} \Delta_a(x - \beta)\, d\beta\right]\left[\int_{E^n} \delta_a(y - \gamma)\, d\gamma\right]$$
$$= 1.$$

Therefore,

$$\int_{E^k \times D} \Gamma_a(\theta - \alpha)\Delta_a(x - \beta)\delta_a(y - \gamma)\, d\alpha\, d\beta\, d\gamma = 1. \tag{3.12}$$

Pick a so that

$$0 < a < \min(a(D_0), \mu/(3\lambda)).$$

This value of a will be used for the rest of the proof. From (3.10) and (3.11), for any $(x, y) \in D_0$,

$$\begin{aligned} &\frac{\partial u}{\partial t}(t, \theta, x, y) + \sum_{p=1}^{k} \frac{\partial u}{\partial \theta_p}(t, \theta, x, y) \\ &\quad = \int_{E^k \times D} \Gamma_a(\theta - \alpha)\Delta_a(x - \beta)\delta_a(y - \gamma)W(t, \alpha, \gamma, 0)\, d\alpha\, d\beta\, d\gamma, \end{aligned} \tag{3.13}$$

and from (3.12),

$$\begin{aligned} &\left|\frac{\partial u}{\partial t}(t, \theta, x, y) + \sum_{p=1}^{k} \frac{\partial u}{\partial \theta_p}(t, \theta, x, y) - W(t, \theta, x, y, 0)\right| \\ &\quad \leqq \int_{E^k \times D} \Gamma_a(\theta - \alpha)\Delta_a(x - \beta)\delta_a(y - \gamma) \\ &\quad\quad \cdot |W(t, \alpha, \beta, \gamma, 0) - W(t, \theta, x, y, 0)|\, d\alpha\, d\beta\, d\gamma. \end{aligned} \tag{3.14}$$

Taking $|\alpha - \theta| \leqq a$, $|\beta - x| \leqq a$, and $|y - \gamma| \leqq a$, we then have

$$\begin{aligned} &|W(t, \alpha, \beta, \gamma, 0) - W(t, \theta, x, y, 0)| \\ &\quad \leqq \lambda(\varepsilon)\{|\alpha - \theta| + |\beta - x| + |\gamma - y|\} \\ &\quad \leqq 3\lambda(\varepsilon)a. \end{aligned}$$

Since $\lambda(\varepsilon)$ is bounded for $\varepsilon \in (0, \varepsilon_0]$ then there exists a bound λ such that $\lambda(\varepsilon) \leqq \lambda$.

Therefore, for $(t, \theta, x, y) \in E \times E^k \times D$, (3.14) becomes

$$\begin{aligned} &\left|\frac{\partial u}{\partial t}(t, \theta, x, y) + \sum_{p=1}^{k} \frac{\partial u}{\partial \theta_p}(t, \theta, x, y) \right. \\ &\quad \left. - W(t, \theta, x, y, 0)\right| \leqq 3\lambda a < \mu. \end{aligned} \tag{3.15}$$

Introduce the function

$$F(\varepsilon) = \sup_{0 \leqq s \leqq L} sf(s/\varepsilon), \tag{3.16}$$

where $f(t)$ was chosen for inequality (3.5). $F(\varepsilon)$ has the property

$$F(\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.17}$$

Let $\varepsilon \in (0, \varepsilon_0]$ and choose $t \in [0, L/\varepsilon]$. Then from (3.16),

$$|\varepsilon u(t, \theta, x, y)| \leqq \varepsilon t f(t) \leqq F(\varepsilon) \tag{3.18}$$

for all θ and $(x, y) \in D_0$. Furthermore, from (3.8),

$$\left\|\varepsilon \frac{\partial u}{\partial \theta}\right\| \leqq k\varepsilon t f(t) I_a^k \leqq k I_a^k F(\varepsilon),$$

$$\left\|\varepsilon \frac{\partial u}{\partial x}\right\| \leqq m\varepsilon t f(t) I_a^m \leqq m I_a^m F(\varepsilon),$$

$$\left\|\varepsilon \frac{\partial u}{\partial y}\right\| \leqq n\varepsilon t f(t) I_a^n \leqq n I_a^n F(\varepsilon).$$

Letting

$$G(\varepsilon) = (1 + k I_a^k + m I_a^m + n I_n^a) F(\varepsilon), \tag{3.19}$$

the conclusion follows.

The main theorem can now be proved.

THEOREM 3.5. *Let $\mu > 0$ be a given constant, and $(\phi(t, \varepsilon), \xi(t, \varepsilon), \eta(t, \varepsilon))$ be a solution of* (3.1) *defined for all $t \geqq 0$ such that there exist constants $\rho > 0$, $L > 0$ with the property that $S_{m+n}((\xi(t, \varepsilon), \eta(t, \varepsilon)); \rho) \subset D = U^m \times V^n$ for $0 \leqq t \leqq L/\varepsilon$ and $0 < \varepsilon \leqq \varepsilon_0$. Then there exists $\varepsilon^* > 0$ such that if $0 < \varepsilon \leqq \varepsilon^*$ and $(\theta(t, \varepsilon), x(t, \varepsilon), y(t, \varepsilon))$ is the solution of* (2.7) *with $\theta(0, \varepsilon) = \phi(0, \varepsilon)$, $x(0, \varepsilon) = \xi(0, \varepsilon)$, $y(0, \varepsilon) = \eta(0, \varepsilon)$, then for $0 \leqq t \leqq L/\varepsilon$,*

$$|\phi(t, \varepsilon) - \theta(t, \varepsilon)| < \mu, \qquad |\xi(t, \varepsilon) - x(t, \varepsilon)| < \mu, \qquad |\eta(t, \varepsilon) - y(t, \varepsilon)| < \mu.$$

Proof. Define

$$\begin{aligned} \Theta^*(t, \theta, x, y) &= \Theta(t, \theta, x, y, 0) - \Theta_0(x, y), \\ X^*(t, \theta, x, y) &= X(t, \theta, x, y, 0) - X_0(x, y). \end{aligned} \tag{3.20}$$

Then for any $(x, y) \in D$,

$$\Theta_0^*(x, y) = X_0^*(x, y) = 0. \tag{3.21}$$

Let

$$v \leqq \frac{\lambda \mu}{8\, e^{2\lambda L}} \tag{3.22}$$

and let D_0 be a compact subset of the domain D.

One may conclude from Lemma 3.4 that there exist functions $u_1(t, \theta, x, y)$, $u_2(t, \theta, x, y)$ and $G(\varepsilon)$ with $G(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that if $0 < \varepsilon \leqq \varepsilon_0$ and $0 \leqq t \leqq L/\varepsilon$, then for $\theta \in E^k$ and $(x, y) \in D_0$:

(a) $u_1(0, \theta, x, y) = u_2(0, \theta, x, y) = 0$;

(b) The first partials of u_i, $i = 1, 2$, with respect to t, θ, x, y exist and

$$\left|\frac{\partial u_1}{\partial t}(t, \theta, x, y) + \sum_{p=1}^{k} \frac{\partial u_1}{\partial \theta_p}(t, \theta, x, y) - \Theta^*(t, \theta, x, y)\right| \leqq v,$$
$$\left|\frac{\partial u_2}{\partial t}(t, \theta, x, y) + \sum_{p=1}^{k} \frac{\partial u_2}{\partial \theta_p}(t, \theta, x, y) - X^*(t, \theta, x, y)\right| \leqq v; \tag{3.23}$$

(c) For $i = 1, 2$,

$$|\varepsilon u_i|, \left\|\varepsilon \frac{\partial u_i}{\partial \theta}\right\|, \left\|\varepsilon \frac{\partial u_i}{\partial x}\right\|, \left\|\varepsilon \frac{\partial u_i}{\partial y}\right\| \leqq G(\varepsilon).$$

Introduce a local transformation

$$\begin{aligned} \bar{\theta} &= \bar{\theta}(t, \varepsilon) = \phi(t, \varepsilon) + \varepsilon u_1(t, \phi(t, \varepsilon), \xi(t, \varepsilon), \eta(t, \varepsilon)), \\ \bar{x} &= \bar{x}(t, \varepsilon) = \xi(t, \varepsilon) + \varepsilon u_2(t, \phi(t, \varepsilon), \xi(t, \varepsilon), \eta(t, \varepsilon)), \\ \bar{y} &= \bar{y}(t, \varepsilon) = \eta(t, \varepsilon), \end{aligned} \tag{3.24}$$

where $(\phi(t, \varepsilon), \xi(t, \varepsilon), \eta(t, \varepsilon))$ is the solution of system (3.1) with $S_{m+n}((\xi(t, \varepsilon), \eta(t, \varepsilon)); \rho) \subset D$ for $t \in [0, L/\varepsilon]$. For the rest of the proof the argument t and parameter ε will be suppressed in the functions ϕ, ξ, η, $\bar{\theta}$, $\bar{x}$ and $\bar{y}$.

Form the new system

$$\begin{aligned} &\frac{d\bar{\theta}}{dt} - d(\varepsilon) - \varepsilon\theta(t, \bar{\theta}, \bar{x}, \bar{y}, \varepsilon) = R_1, \\ &\frac{d\bar{x}}{dt} - \varepsilon X(t, \bar{\theta}, \bar{x}, \bar{y}, \varepsilon) = R_2, \\ &\frac{d\bar{y}}{dt} - A\bar{y} - \varepsilon Y(t, \bar{\theta}, \bar{x}, \bar{y}, \varepsilon) = R_3. \end{aligned} \tag{3.25}$$

Since (ϕ, ξ, η) satisfy (3.1) one can compute the error terms R_1, R_2, R_3. Using (3.24) and (3.25),

$$\begin{aligned} R_1 = \varepsilon&\left\{\frac{\partial u_1}{\partial t} + \sum_{p=1}^{k} \frac{\partial u_1}{\partial \phi_p} - \Theta^*(t, \phi, \xi, \eta)\right\} \\ &+ \varepsilon\frac{\partial u_1}{\partial \phi}[d(\varepsilon) - 1] + \varepsilon^2 \frac{\partial u_1}{\partial \phi}\Theta_0(\xi, \eta) \\ &+ \varepsilon^2 \frac{\partial u_1}{\partial \xi} X_0(\xi, \eta) + \varepsilon \frac{\partial u_1}{\partial \eta} A\eta + \varepsilon^2 \frac{\partial u_1}{\partial \eta} \; Y(t, \phi, \xi, \eta, \varepsilon) \\ &+ \varepsilon(\Theta(t, \theta, \xi, \eta, 0) - \Theta(t, \phi + \varepsilon u_1, \xi + \varepsilon u_2, \eta, 0)\} \\ &+ \varepsilon\{\Theta(t, \phi + \varepsilon u_1, \xi + \varepsilon u_2, \eta, 0) \\ &\quad - \Theta(t, \phi + \varepsilon u_1, \xi + \varepsilon u_2, \eta, \varepsilon)\}, \end{aligned} \tag{3.26}$$

(continued)

$$R_2 = \varepsilon\left\{\frac{\partial u_2}{\partial t} + \sum_{p=1}^{k}\frac{\partial u_2}{\partial \phi_p} - X^*(t,\phi,\xi,\eta)\right\}$$

$$+ \varepsilon\frac{\partial u_2}{\partial \phi}[d(\varepsilon) - 1] + \varepsilon^2\frac{\partial u_2}{\partial \phi}\Theta_0(\xi,\eta)$$

(3.26) (continued)

$$+ \varepsilon^2\frac{\partial u_2}{\partial \xi}X_0(\xi,\eta) + \varepsilon\frac{\partial u_2}{\partial \eta}A\eta$$

$$+ \varepsilon^2\frac{\partial u_2}{\partial \eta}Y(t,\phi,\xi,\eta,\varepsilon)$$

$$+ \varepsilon\{X(t,\phi,\xi,\eta,0) - X(t,\phi + \varepsilon u_1,\xi + \varepsilon y_2,\eta,0)\}$$

$$+ \varepsilon\{X(t,\phi + \varepsilon u_1,\xi + \varepsilon u_2,\eta,0)$$

$$+ X(t,\phi + \varepsilon u_1,\xi + \varepsilon u_2,\eta,\varepsilon)\},$$

$$R_3 = \varepsilon\{Y(t,\phi,\xi,\eta,\varepsilon) - Y(t,\phi + \varepsilon u_1,\xi + \varepsilon u_2,\eta,\varepsilon)\}.$$

Let M be the bound for $\Theta, X, Y, \Theta_0, X_0$ and λ the Lipschitz constant. From the uniform continuity of Θ and X in Σ there exists a function $h(\varepsilon)$ such that $h(\varepsilon) \to 0$ as $\varepsilon \to 0$ and

(3.27)
$$|\Theta(t,\theta,x,y,0) - \Theta(t,\theta,x,y,\varepsilon)| \leqq h(\varepsilon),$$
$$|X(t,\theta,x,y,0) - X(t,\theta,x,y,\varepsilon)| \leqq h(\varepsilon).$$

Therefore, for $\varepsilon \in (0,\varepsilon_0]$ and $0 \leqq t \leqq L/\varepsilon$,

$$|R_1| \leqq \varepsilon\left|\frac{\partial u_1}{\partial t} + \sum_{p=1}^{k}\frac{\partial u_1}{\partial \theta_p} - \Theta^*(t,\phi,\xi,\eta)\right|$$

$$+ \varepsilon\left\|\frac{\partial u_1}{\partial \phi}\right\| |d(\varepsilon) - 1| + \varepsilon^2\left\|\frac{\partial u_1}{\partial \phi}\right\|$$

$$\cdot |\Theta_0(\xi,\eta)| + \varepsilon^2\left\|\frac{\partial u_1}{\partial \xi}\right\| |X_0(\xi,\eta)|$$

$$+ \varepsilon\left\|\frac{\partial u_1}{\partial \eta}\right\| \|A\| |\eta|$$

$$+ \varepsilon^2\left\|\frac{\partial u_1}{\partial \eta}\right\| |Y(t,\phi,\xi,\eta,\varepsilon)| + \varepsilon|\Theta(t,\phi,\xi,\eta,0)$$

$$- \Theta(t,\phi + \varepsilon u_1,\xi + \varepsilon u_2,\eta,0)| + \varepsilon|\Theta(t,\phi + \varepsilon u_1,$$

$$\xi + \varepsilon u_2,\eta,0) - \Theta(t,\phi + \varepsilon u_1,\xi + \varepsilon u_2,\eta,\varepsilon)|,$$

and therefore from (3.23) and (3.27),

(3.28)
$$|R_1| \leqq \varepsilon v + \{2\lambda\varepsilon + |d(\varepsilon) - 1| + 3\varepsilon M\}G(\varepsilon) + \varepsilon h(\varepsilon)$$
$$+ \varepsilon\left\|\frac{\partial u_1}{\partial \eta}\right\| \|A\| |\eta|.$$

Since the eigenvalues of A have negative real parts $\|e^{At}\| \leqq c\, e^{-bt}$ for some constants $b, c > 0$, and from the third equation of (3.1) and Lemma 3.1,

$$|\eta| \leqq k_0\, e^{-bt} + \frac{\varepsilon M c}{b}, \tag{3.29}$$

where $k_0 = |\eta(0)|c$. Introducing (3.29) into (3.28) yields

$$\begin{aligned} |R_1| &\leqq \varepsilon\nu + \{2\lambda\varepsilon + |d(\varepsilon) - 1| + 3\varepsilon M\}G(\varepsilon) + \varepsilon h(\varepsilon) \\ &\quad + \varepsilon\left\|\frac{\partial u_1}{\partial \eta}\right\| \|A\| \left\{k_0\, e^{-bt} + \frac{\varepsilon c M}{b}\right\}. \end{aligned} \tag{3.30}$$

$|R_2|$ and $|R_3|$ can be estimated in a similar manner. Taking norms one gets for $0 \leqq t \leqq L/\varepsilon$,

$$\begin{aligned} |R_2| &\leqq \varepsilon\left|\frac{\partial u_2}{\partial t} + \sum_{p=1}^{k} \frac{\partial u_2}{\partial \phi_p} - X^*(t, \phi, \xi, \eta)\right| \\ &\quad + \varepsilon\left\|\frac{\partial u_2}{\partial \phi}\right\| |d(\varepsilon) - 1| + \varepsilon^2\left\|\frac{\partial u_2}{\partial \theta}\right\| M \\ &\quad + \varepsilon^2\left\|\frac{\partial u_2}{\partial \xi}\right\| M + \varepsilon\left\|\frac{\partial u_2}{\partial \eta}\right\| \|A\|\, |\eta| + \varepsilon^2\left\|\frac{\partial u_2}{\partial \eta}\right\| M \\ &\quad + \varepsilon^2\lambda|u_1| + \varepsilon|X(t, \phi + \varepsilon u_1, \xi + \varepsilon u_2, \eta, 0) \\ &\qquad\qquad - X(t, \phi + \varepsilon u_1, \xi + \varepsilon u_2, \eta, \varepsilon)| \\ &\leqq \varepsilon\nu + \{2\lambda\varepsilon + |d(\varepsilon) - 1| + 3\varepsilon M\}G(\varepsilon) + \varepsilon h(\varepsilon) \\ &\quad + \varepsilon\left\|\frac{\partial u_2}{\partial \eta}\right\| \|A\| \left\{k_0\, e^{-bt} + \frac{\varepsilon c M}{b}\right\}. \end{aligned} \tag{3.31}$$

Finally, estimating $|R_3|$ for $0 \leqq t \leqq L/\varepsilon$,

$$|R_3| \leqq \varepsilon|Y(t, \phi, \xi, \eta, \varepsilon) - Y(t, \phi + \varepsilon u_1, \xi + \varepsilon u_2, \eta, \varepsilon)| \leqq 2\varepsilon\lambda G(\varepsilon). \tag{3.32}$$

Let $(\theta_0, x_0, y_0) \in E^k \times D$ and $(\theta(t), x(t), y(t))$ be a solution of (2.7) such that $\theta(0, \varepsilon) = \phi(0, \varepsilon) = \theta_0, x(0, \varepsilon) = \xi(0, \varepsilon) = x_0$ and $y(0, \varepsilon) = \eta(0, \varepsilon) = y_0$. Then, by continuity, there is certainly some interval $0 \leqq t \leqq t^*$, with $t^* \leqq L/\varepsilon$, such that $(x(t), y(t)) \in D$ on $[0, t^*]$. Eventually it will be shown that $t^* = L/\varepsilon$. For the moment, let $t \in [0, t^*]$ and consider the system representing the difference between (2.7) and (3.25):

$$\begin{aligned} \frac{d}{dt}(\bar\theta - \theta) &= \varepsilon[\Theta(t, \bar\theta, \bar x, \bar y, \varepsilon) - \Theta(t, \theta, x, y, \varepsilon)] + R_1, \\ \frac{d}{dt}(\bar x - x) &= \varepsilon[X(t, \bar\theta, \bar x, \bar y, \varepsilon) - X(t, \theta, x, y, \varepsilon)] + R_2, \\ \frac{d}{dt}(\bar y - y) &= A(\bar y - y) + \varepsilon[Y(t, \bar\theta, \bar x, \bar y, \varepsilon) - Y(t, \theta, x, y, \varepsilon)] + R_3. \end{aligned} \tag{3.33}$$

For $t \in [0, t^*]$,

$$\left|\frac{d}{dt}(\bar{\theta} - \theta)\right| \leqq \varepsilon\lambda\{|\bar{\theta} - \theta| + |\bar{x} - x| + |\bar{y} - y|\} + |R_1|,$$

$$\left|\frac{d}{dt}(\bar{x} - x)\right| \leqq \varepsilon\lambda\{|\bar{\theta} - \theta| + |\bar{x} - x| + |\bar{y} - y|\} + |R_2|, \tag{3.34}$$

$$\frac{d}{dt}(\bar{y} - y) = A(\bar{y} - y) + \varepsilon[Y(t, \bar{\theta}, \bar{x}, \bar{y}, \varepsilon) - Y(t, \theta, x, y, \varepsilon)] + R_3,$$

where λ is the Lipschitz constant for Θ, X. From (3.24), $\bar{y}(0) = \eta(0) = y(0) = y_0$. Therefore on the interval $[0, t^*]$ the variation of parameters formula gives

$$\begin{aligned}
|\bar{y} - y| &\leqq \varepsilon \int_0^t \|e^{A(t-s)}\| \, |Y(s, \bar{\theta}, \bar{x}, \bar{y}, \varepsilon) - Y(s, \theta, x, y, \varepsilon)| \, ds \\
&\quad + \varepsilon \int_0^t \|e^{A(t-s)}\| \, |R_3(s)| \, ds \\
&\leqq 2\varepsilon c M \int_0^t e^{-b(t-s)} \, ds + \varepsilon c \int_0^t e^{-b(t-s)} |R_3(s)| \, ds \\
&\leqq \frac{2c\varepsilon M}{b} + \frac{2\varepsilon^2 \lambda c L G(\varepsilon)}{b},
\end{aligned} \tag{3.35}$$

since $|Y| \leqq M$.

Therefore, for $0 \leqq t \leqq t^*$, (3.34) reduces to

$$\begin{aligned}
\left|\frac{d}{dt}(\bar{\theta} - \theta)\right| &\leqq \varepsilon\lambda\{|\bar{\theta} - \theta| + |\bar{x} - x|\} + \varepsilon\lambda\left\{\frac{2\varepsilon c M}{b} + \frac{2\lambda c L G(\varepsilon)}{b}\varepsilon^2\right\} \\
&\quad + \{(2\lambda + 3M)\varepsilon + |d(\varepsilon) - 1|\}G(\varepsilon) \\
&\quad + \varepsilon h(\varepsilon) + \varepsilon\left\|\frac{\partial u_1}{\partial \eta}\right\| \|A\| \left\{k_0 e^{-bt} + \frac{\varepsilon c M}{b}\right\} + \varepsilon\nu,
\end{aligned} \tag{3.36}$$

$$\begin{aligned}
\left|\frac{d}{dt}(\bar{x} - x)\right| &\leqq \varepsilon\lambda\{|\bar{\theta} - \theta| + |\bar{x} - x|\} + \varepsilon\lambda\left\{\frac{2\varepsilon c M}{b} + \frac{2\lambda c L G(\varepsilon)}{b}\varepsilon^2\right\} \\
&\quad + \{(2\lambda + 3M)\varepsilon + |d(\varepsilon) - 1|\}G(\varepsilon) + \varepsilon h(\varepsilon) \\
&\quad + \varepsilon\left\|\frac{\partial u_2}{\partial \eta}\right\| \|A\| \left\{k_0 e^{-bt} + \frac{\varepsilon c M}{b}\right\} + \varepsilon\nu.
\end{aligned}$$

In order to estimate the size of $|\bar{\theta} - \theta|$ and $|\bar{x} - x|$ consider a new vector formed by

$$\begin{pmatrix} \bar{\theta} - \theta \\ \bar{x} - x \end{pmatrix}$$

and define

$$\left|\begin{pmatrix} \bar{\theta} - \theta \\ \bar{x} - x \end{pmatrix}\right| = |\bar{\theta} - \theta| + |\bar{x} - x|.$$

Then for $t \in [0, t^*]$,

$$\left|\frac{d}{dt}\begin{pmatrix}\bar\theta - \theta \\ \bar x - x\end{pmatrix}\right| \leqq 2\varepsilon\lambda\left\|\begin{pmatrix}\bar\theta - \theta \\ \bar x - x\end{pmatrix}\right\| + 2\varepsilon\lambda\left\{\frac{2\varepsilon cM}{b} + \frac{2\lambda cLG(\varepsilon)\varepsilon^2}{b}\right\}$$

$$(3.37) \qquad + 2\{(2\lambda + 3M)\varepsilon + |d(\varepsilon) - 1|\}G(\varepsilon) + 2\varepsilon h(\varepsilon)$$

$$+ \varepsilon\|A\|\left\{k_0 e^{-bt} + \frac{\varepsilon Mc}{b}\right\}\left\{\left\|\frac{\partial u_1}{\partial\eta}\right\| + \left\|\frac{\partial u_2}{\partial\eta}\right\|\right\} + 2\varepsilon v.$$

Solving the differential inequality (3.37), keeping in mind that $\bar\theta(0) = \phi(0)$, $\theta(0) = \bar x(0) = \xi(0) = x(0)$, one gets, for $t \in [0, t^*]$ and $0 < \varepsilon \leqq \varepsilon_0$,

$$\left\|\begin{pmatrix}\bar\theta - \theta \\ \bar x - x\end{pmatrix}\right\| \leqq \int_0^t e^{2\varepsilon\lambda(t-s)}\left[2\varepsilon\lambda\left\{\frac{2\varepsilon cM}{b} + \frac{2\lambda cLG(\varepsilon)\varepsilon^2}{b}\right\} + 2\varepsilon v\right.$$

$$+ 2\{(2\lambda + 3M)\varepsilon + |d(\varepsilon) - 1|\}G(\varepsilon) + 2\varepsilon h(\varepsilon)$$

$$\left. + \varepsilon\|A\|\left\{k_0 e^{-bs} + \frac{\varepsilon cM}{b}\right\}\left\{\left\|\frac{\partial u_1}{\partial\eta}\right\| + \left\|\frac{\partial u_2}{\partial\eta}\right\|\right\}\right] ds$$

$$(3.38) \qquad \leqq \left[\frac{2e^{2\lambda L}cM}{b}\right]\varepsilon + \left[\frac{e^{2\lambda L}}{\lambda}\right]v + \left[\frac{e^{2\lambda L}}{\lambda}\right]h(\varepsilon)$$

$$+ e^{2\lambda L}\left[\frac{2\lambda cL\varepsilon^2}{b} + \frac{1}{\lambda}(2\lambda + 3M)\right.$$

$$\left. + \frac{|d(\varepsilon) - 1|}{\lambda\varepsilon} + \frac{2\|A\|k_0}{b} + \frac{\|A\|cM}{b\lambda}\right]G(\varepsilon).$$

But v was chosen in (3.22) so that $(e^{2\lambda L}/\lambda)v < \mu/8$, thus pick ε^* so that if $0 \leqq \varepsilon \leqq \varepsilon^*$, then

$$\left[\frac{2e^{2\lambda L}cM}{b}\right]\varepsilon + \left[\frac{e^{2\lambda L}}{\lambda}\right]h(\varepsilon) + e^{2\lambda L}\left[\frac{2\lambda cL\varepsilon^2}{b} + \frac{1}{\lambda}(2\lambda + 3M)\right.$$

$$\left. + \frac{|d(\varepsilon) - 1|}{\lambda\varepsilon} + \frac{2\|A\|k_0}{b} + \frac{\|A\|cM}{b\lambda}\right]G(\varepsilon) < \frac{\mu}{8}$$

and

$$G(\varepsilon) < \frac{\mu}{4} \quad \text{as well as} \quad \frac{2\varepsilon cM}{b} + \frac{2\lambda cLG(\varepsilon)\varepsilon^2}{b} < \frac{\mu}{4}.$$

Then for $0 < \varepsilon \leqq \varepsilon^*$, $t \in [0, t^*]$,

$$(3.39) \qquad |\bar\theta - \theta| \leqq \frac{\mu}{4}, \quad |\bar y - y| \leqq \frac{\mu}{4}, \quad |\bar x - x| \leqq \frac{\mu}{4},$$

and considering (3.22),

$$|\theta - \phi| \leqq |\theta - \bar\theta| + |\bar\theta - \phi| < \mu.$$

Similarly $|x - \xi| < \mu$, $|y - \eta| < \mu$, and the conclusion holds.

One can now show that $t^* = L/\varepsilon$. Let

$$t' = \text{l.u.b. } \{t \mid |\phi - \theta| \leqq \mu, |\xi - x| \leqq \mu, |\eta - y| \leqq \mu\}.$$

If $t' = L/\varepsilon$ then the proof is finished since one could have $t^* = t' = L/\varepsilon$ above. Suppose that $t' < L/\varepsilon$. Then on the interval $[0, t']$, one difference, say $|\phi - \theta|$, must satisfy $|\phi - \theta| < \mu$ for $t \in [0, t')$ and $|\phi(t') - \theta(t')| = \mu$. But, for $0 < \varepsilon \leqq \varepsilon^*$, $G(\varepsilon) < \mu/2$. Let $\omega = \frac{1}{2}\{(\mu/2) - G(\varepsilon)\}$ and note $\omega > 0$. Then

$$\mu = |\phi(t') - \theta(t')| \leqq |\bar{\theta}(t') - \theta(t')| + |\varepsilon U_1| \leqq \frac{\mu}{4} + G(\varepsilon)$$

$$\leqq \frac{\mu}{4} + \frac{\mu}{2} - 2\omega < \mu,$$

which is a contradiction. Thus take $t^* = t' = L/\varepsilon$.

Bogoliubov and Mitropolsky [2, p. 429] proved a version of Theorem 3.5 under the more restrictive assumption that the solution of (3.1) lies with its ρ-neighborhood inside D for all $t > 0$.

4. Domains of stability. Consider the following system:

$$\frac{dx}{dt} = \varepsilon X_0(x, y), \qquad \frac{dy}{dt} = Ay. \tag{4.1}$$

From hypothesis H4 system (4.1) has an asymptotically stable equilibrium point $(x_0, 0)$. From Theorem 2.2 there exists an asymptotically stable integral manifold for (2.7). Let Δ_0 be the domain of stability of the integral manifold $x = f(t, \theta, \varepsilon)$, $y = g(t, \theta, \varepsilon)$ of (2.7). Furthermore Theorem 2.2 implies that Δ_0 is not empty. The following result is an analogue of one due to Loud and Sethna [11] and gives a tool to estimate Δ_0.

For the results of this section we will make the restriction that $D = U^m \times V^n$ has a compact closure. D, for example, could be a large bounded set in E^{m+n}.

THEOREM 4.1. *Let D_0 be the domain of stability of $(x_0, 0)$ for system (4.1), where $D_0 \subset D$. Let $C \subset D_0$ be a closed set and $O \subset C$ an open set with the property that, if $(x(t), y(t))$ is a solution of (4.1) such that $(x(0), y(0)) \in O$, then $(x(t), y(t)) \notin \partial C$ for $t \geqq 0$. Then for ε sufficiently small $E^k \times O \subset \Delta_0$.*

Proof. Let σ_0 be given by Theorem 2.2. Let $(\theta_1, x_1, y_1) \in E^k \times O$ be arbitrary. Let $(\xi(t), \eta(t))$ be the solution of (4.1) such that $\xi(0) = x_1$, $\eta(0) = y_1$ and $\phi(t)$ the solution of

$$\frac{d\phi}{dt} = d(\varepsilon) + \varepsilon\theta_0(\xi(t), \eta(t))$$

such that $\phi(0) = \theta_1$. Then $(\phi(t), \xi(t), \eta(t))$ is the unique solution of

$$\begin{aligned} \frac{d\theta}{dt} &= d(\varepsilon) + \varepsilon\theta_0(x, y), \\ \frac{dx}{dt} &= \varepsilon X_0(x, y), \\ \frac{dy}{dt} &= Ay, \end{aligned} \tag{4.2}$$

such that $\phi(0) = \theta_1$, $\xi(0) = x_1$, $\eta(0) = y_1$.

From the hypotheses $(\xi(t), \eta(t)) \notin \partial C$ and, since $O \subset D_0$, there is a set $S_{m+n}((x_0, 0); \rho_0) \subset O$ such that $(\xi(t), \eta(t)) \in S_{m+n}((x_0, 0); \rho_0)$ for $t \geqq t_1$ for some $t_1 > 0$. One can then choose a uniform ρ^*-neighborhood of $(\xi(t), \eta(t))$ for $t \geqq 0$ contained in O.

By introducing a change of variables, $t \to \varepsilon t$ in (4.1), and using H4 it is easy to find an $L > 0$ such that if $0 < \varepsilon \leqq \varepsilon_0$, then for $t > L/2\varepsilon$,

$$|\xi(t) - x_0| < \frac{\sigma_0}{3}, \qquad |\eta(t)| < \frac{\sigma_0}{3}, \tag{4.3}$$

using the fact that $(x_0, 0)$ is asymptotically stable. Now choose $\varepsilon_1 \leqq \varepsilon_0$, $\varepsilon_1 > 0$, such that if $0 < \varepsilon \leqq \varepsilon_1$ then, if M is the uniform bound for Y, as before,

$$\varepsilon\left(\frac{Mc\, e^{\lambda L}}{b}\right) \leqq \min\left(\frac{\sigma_0}{3}, \frac{\rho^*}{4}\right), \tag{4.4}$$

where b and c have been chosen positive constants so that $\|e^{At}\| \leqq c\, e^{-bt}$, for $t \geqq 0$. This is always possible by assumption H4.

Let $\varepsilon \in (0, \varepsilon_1]$ and $(\psi(t), u(t), v(t))$ be the solution of system (3.1) such that $\psi(0) = \theta_1, u(0) = x_1, v(0) = y_1$. Then consider

$$\begin{aligned} &\frac{d}{dt}(\psi - \phi) = \varepsilon[\Theta_0(u, v) - \Theta_0(\xi, \eta)], \\ &\frac{d}{dt}(u - \xi) = \varepsilon[X_0(u, v) - X_0(\xi, \eta)], \\ &\frac{d}{dt}(v - \eta) = A(v - \eta) + \varepsilon Y(t, \psi, u, v, \varepsilon). \end{aligned} \tag{4.5}$$

Since Y is uniformly bounded by M one has from Lemma 3.1,

$$|v - \eta| \leqq \frac{\varepsilon c M}{b} \leqq \min\left(\frac{\sigma_0}{3}, \frac{\rho^*}{4}\right) \quad \text{for } 0 \leqq t \leqq \frac{L}{\varepsilon}$$

and then,

$$\begin{aligned} |u - \xi| &\leqq \frac{\varepsilon^2 c \lambda M}{b} \int_0^t e^{\varepsilon\lambda(t-s)}\, ds \leqq \frac{\varepsilon c M}{b} e^{\lambda L} \\ &< \min\left(\frac{\sigma_0}{3}, \frac{\rho^*}{4}\right) \end{aligned} \tag{4.6}$$

for all $t \in [0, L/\varepsilon]$, where $0 < \varepsilon \leqq \varepsilon_1$. Thus, for $0 \leqq t \leqq L/\varepsilon$, $(u(t), v(t)) \in D$ along with its $\rho^*/4$-neighborhood provided $0 < \varepsilon \leqq \varepsilon_1$.

By Theorem 3.5 choose $\varepsilon^* \leqq \varepsilon_1$ such that if $(\theta(t), x(t), y(t))$ is the solution of system (2.7) with $\theta(0) = \theta_1$, $x(0) = x_1$, $y(0) = y$, then for $0 \leqq t \leqq L/\varepsilon$, $\varepsilon \in (0, \varepsilon^*]$,

$$|\theta(t) - \psi(t)| \leqq \frac{\sigma_0}{3}, \qquad |x(t) - u(t)| \leqq \frac{\sigma_0}{3}, \qquad |y(t) - v(t)| \leqq \frac{\sigma_0}{3}. \tag{4.7}$$

Let $0 < \varepsilon \leqq \varepsilon^*$ and t^* be chosen so that $(L/2\varepsilon) < t^* < L/\varepsilon$. From (4.3), (4.6), and (4.7),

$$\begin{aligned} |x(t^*) - x_0| &\leqq |x(t^*) - u(t^*)| + |u(t^*) - \xi(t^*)| + |\xi(t^*) - x_0| < 3(\sigma_0/3) = \sigma_0, \\ |y(t^*)| &\leqq |y(t^*) - v(t^*)| + |v(t^*) - \eta(t^*)| + |\eta(t^*)| < 3(\sigma_0/3) = \sigma_0. \end{aligned} \tag{4.8}$$

Then Theorem 2.2 yields

$$|x(t) - f(t, \theta(t), \varepsilon)| \to 0, \qquad |y(t) - g(t, \theta(t), \varepsilon)| \to 0$$

as $t \to \infty$, provided $0 < \varepsilon \leqq \varepsilon^*$. Therefore $(\theta_1, x_1, y_1) \in \Delta_0$ or $E^k \times O \subset \Delta_0$.

In order to apply this theorem to specific situations one must be able to determine the sets D_0, C and O with the required properties. The next results will show a method of computing these sets based upon Liapunov functions. The results and methods are analogous to those of Loud and Sethna [11] and, in particular, if system (2.7) does not have the θ and y equations the results are identical to those of Sethna and Moran [13].

Returning to system (4.1), introduce the change of variable $w = x - x_0$ and rewrite (4.1) in the form

$$\frac{dw}{dt} = \varepsilon C w + \varepsilon\{X_0(w + x_0, y) - Cw\}, \qquad \frac{dy}{dt} = Ay, \tag{4.9}$$

where $C = (\partial X_0/\partial x)(x_0, 0), (x, y) \in D$.

For later reference we will introduce the notation $\dot{V}_{(4.1)}(x, y)$ to mean $\dot{V}_{(4.1)}(x, y) = \operatorname{grad} V(x, y) \cdot (\dot{x}, \dot{y})$, where $\dot{x}$ and $\dot{y}$ are given by system (4.1). The subscript (4.1) refers to the system along whose solution the time derivative of V is being considered.

THEOREM 4.2. *Let there exist a continuous scalar function $V(w, y)$ for* (4.9) *such that $V(0, 0) = \dot{V}_{(4.9)}(0, 0) = 0$, $V(w, y) > 0$ and $\dot{V}_{(4.9)}(w, y) < 0$ for $(w, y) \neq (0, 0)$ in Ω_l where $\Omega_l = \{(w, y) | V(w, y) < l\}$. Let $D_0(l) = \{(w + x_0, y) | (w, y) \in \Omega_l\}$. Define for η, an arbitrary parameter, such that* $0 < \eta < l$,

$$C_\eta = \{(x, y) | (x, y) \in D, x = x_0 + w, \text{ where } (w, y) \in \overline{\Omega_{l-\eta}}\},$$

where

$$\overline{\Omega_{l-\eta}} = \{(w, y) | V(w, y) \leqq l - \eta\},$$

and

$$O_\eta = \{(x, y) | (x, y) \in D, x = x_0 + w, \text{ where } (w, y) \in \Omega_{l-\eta}\},$$

where

$$\Omega_{l-\eta} = \{(w, y) | V(w, y) < l - \eta\}.$$

Then, for each $\eta \in (0, l)$, the sets $D_0(l)$, C_η and *O_η satisfy the hypotheses of Theorem* 4.1.

Proof. By a fundamental result on stability due to LaSalle and Lefschetz [10, p. 59] the set $D_0(l)$ is contained in the domain of asymptotic stability of $(x_0, 0)$. Then by construction, $C_\eta \subset D_0(l) \cap D$, where η is an arbitrary parameter such that $0 < \eta < l$. Furthermore from the definition above, $O_\eta \subset C_\eta \subset D_0(l) \cap D$.

From the continuity of V, C_η is a closed subset of $D_0(l) \cap D$ and O_η is an open subset of C_η.

Let $(x(t), y(t))$ be a solution of (4.1) such that $(x(t_0), y(t_0)) \in O_\eta$ for some t_0. Then $w(t) = x(t) - x_0$ and $y(t)$ are solutions of (4.9) such that $(w(t_0), y(t_0) \in \Omega_{l-\eta}$. Therefore, since $\dot{V}_{(4.9)}(w, y) < 0$ in Ω_l, $V(w(t), y(t)) \leqq V(w(t_0), y(t_0)) < l - \eta$ for $t \geqq t_0$. Therefore for $t \geqq t_0$, $(x(t), y(t)) \notin \partial C_\eta$.

COROLLARY 4.3. *Let* $O(l) = \bigcup_{0<\eta<l} O_\eta$. *Then* $E^k \times O(l) \subset \Delta_0$, *the domain of stability of the integral manifold for* (2.7).

Proof. Let $(\theta_1, x_1, y_1) \in E^k \times O(l)$. Then there is some $\eta \in (0, l)$ such that $(x_1, y_1) \in O_\eta$. Then $E^k \times O_\eta \subset \Delta_0$. Therefore $E^k \times O(l) \subset \Delta_0$.

The proofs of the previous results have been included for completeness since they are quite analogous to the proofs developed by Loud and Sethna [11], as mentioned above. In the next sections we will apply these results to several examples, which have been discussed from other points of view by Hale [5, pp. 145–169]. In order to do so however we will need to construct Liapunov functions for quasi-linear systems. A method for doing this has been given in Krasovskii [8, p. 87], which we will briefly review here.

Let

$$\dot{x} = Px + \phi(x), \tag{4.10}$$

where P has all its eigenvalues with negative real parts. Choose a negative definite quadratic form

$$W(x) = -x^T C x, \tag{4.11}$$

where x^T is the transpose of x, and C is symmetric and positive definite. Construct

$$V(x) = x^T A x, \tag{4.12}$$

where $A^T = A$ and $P^T A + AP = -C$.

Using (4.12) compute $\dot{V}_{(4.10)}(x)$ as

$$\dot{V}_{(4.10)}(x) = W(x) + \phi(x)^T A x + x^T A \phi(x). \tag{4.13}$$

Let $\phi(x)$ have the special form

$$\phi(x) = H(x)x, \tag{4.14}$$

where $H(x) = (h_{ij}(x))$ for some functions $h_{ij}(x)$ of the vector x. Then one can write

$$\dot{V}_{(4.10)}(x) = -x^T E(x) x, \tag{4.15}$$

where $E(x) = C - H(x)^T A - AH(x)$. The set of values for which (4.15) is negative definite can be determined by using Sylvester's rule of principal minors. That is, for the matrix $E(x)$ to generate a positive quadratic form $-\dot{V}_{(4.10)}(x)$ there must exist $\gamma > 0$ such that the principal minors, Δ_p, satisfy

$$\Delta_n > \gamma, \cdots, \Delta_1 > \gamma \tag{4.16}$$

uniformly in x.

5. van der Pol equation. Consider the system

$$\frac{d^2x}{dt^2} - \varepsilon(1 - x^2)\frac{dx}{dt} + x = 0. \tag{5.1}$$

This system is equivalent to

$$\frac{dx_1}{dt} = x_2, \qquad \frac{dx_2}{dt} = -x_1 + \varepsilon(1 - x_1^2)x_2. \tag{5.2}$$

Introduce the change of coordinates

$$x_1 = \rho\cos\theta, \qquad x_2 = \rho\sin\theta, \tag{5.3}$$

where $\rho \geqq 0$. Then (5.2) becomes

$$\begin{aligned} \frac{d\theta}{dt} &= 1 + \varepsilon(1 - \rho^2\cos^2\theta)\sin\theta\cos\theta, \\ \frac{d\rho}{dt} &= \varepsilon(1 - \rho^2\cos^2\theta)\rho\sin^2\theta. \end{aligned} \tag{5.4}$$

System (5.4) is of type (2.7) with the y equation absent.

To apply the theory begin by taking the domain D to be any large, but bounded, interval containing $\rho = 2$. For the purpose here take $D = [0, \rho_0]$, where $2 \ll \rho_0$. Clearly H1–H3 are satisfied and the averaged system for (5.4) is given by

$$\frac{d\theta}{dt} = 1, \qquad \frac{d\rho}{dt} = \frac{\varepsilon\rho}{2}\left(1 - \frac{\rho^2}{4}\right). \tag{5.5}$$

Now, $(\rho/2)(1 - \rho^2/4) = 0$ iff $\rho = 0$, $\rho = 2$.

The Jacobian of $(\rho/2)(1 - \rho^2/4)$ is $\frac{1}{2} - 3\rho^2/8$. At $\rho = 2$ its value is negative and at $\rho = 0$ its value is positive. Hypothesis H4 is satisfied for $\rho = 2$. Note here that $\rho = 0$ will correspond to the unstable origin for (5.2).

If the change of variables $r = \rho - 2$ is introduced into (5.5), then

$$\frac{d\theta}{dt} = 1, \qquad \frac{dr}{dt} = -\varepsilon r - \frac{\varepsilon r}{8}(6r + r^2), \tag{5.6}$$

where $r \geqq -2$. Then, the second equation of (5.6) is of the form (4.9).

From Theorem 2.2 system (5.4) has a one-parameter, asymptotically stable, integral manifold represented parametrically by $\rho = f(\theta, \varepsilon)$, which satisfies $f(\theta, 0) = 2$. This manifold represents a periodic orbit of (5.2) and therefore of (5.1).

One can use Corollary 4.3 to estimate the domain of stability of $\rho = f(\theta, \varepsilon)$. Consider $v(r) = r^2$. Then $\dot{V}_{(5.6)}(r) = 2r\dot{r} = -(\varepsilon r^2/4)(r + 2) \times (r + 4)$ which satisfies $\dot{V}_{(5.6)}(r) < 0$ iff $r > -2$, $r \neq 0$. Let l be a real number such that $(\rho_0 - 2)^2 < l$. As in Theorem 4.2 take $\Omega_l = \{r \mid v(r) < l\}$. Then $r = 0$ is asymptotically stable, and $\Omega_l = (-2, \sqrt{l})$. As in Theorem 4.2, $D_0(l) = (0, 2 + \sqrt{l})$ and $D_0(l) \cap D = (0, \rho_0]$ and for $\eta \in (0, l)$, $\Omega_{l-\eta} = (-2, \sqrt{l - \eta})$. Therefore $O\eta = (0, \rho_0)$. Thus $O = (0, \rho_0)$ and therefore by Corollary 4.3, $E \times (0, \rho_0) \subset \Delta_0$, where Δ_0 is the domain of stability of the integral manifold $\rho = f(\theta, \varepsilon)$ for (5.4), where ε is sufficiently small.

Transforming back to (5.2), $(x_1(\theta), x_2(\theta))$, $x_1(\theta) = f(\theta, \varepsilon)\cos\theta$, $x_2(\theta) = f(\theta, \varepsilon)\sin\theta$, is an asymptotically stable one-parameter integral manifold of (5.2), periodic in θ of period 2π, and furthermore, for ε suitably restricted, any solution of (5.2) beginning in $\{(x_1, x_2) | 0 < x_1^2 + x_2^2 < \rho_0^2\}$ is attracted to the stable manifold.

6. Forced van der Pol equation. For this example consider the equation

$$\frac{d^2z}{dt^2} - \varepsilon(1 - z^2)\frac{dz}{dt} + z = A\sin\sqrt{2}t + B\sin\sqrt{3}t. \tag{6.1}$$

This is equivalent to the system

$$\begin{aligned} \frac{dz_1}{dt} &= z_2, \\ \frac{dz_2}{dt} &= -z_1 + \varepsilon(1 - z_1^2)z_2 + A\sin\sqrt{2}t + B\sin\sqrt{3}t, \end{aligned} \tag{6.2}$$

where $\varepsilon > 0$, A and B are parameters. Introduce the following change of coordinates:

$$\begin{aligned} z_1 &= x_1\cos t + x_2\sin t - A\sin\sqrt{2}t - (B/2)\sin\sqrt{3}t, \\ z_2 &= -x_1\sin t + x_2\cos t - A\sqrt{2}\cos\sqrt{2}t - (B\sqrt{3}/2)\cos\sqrt{3}t. \end{aligned} \tag{6.3}$$

Then through this change of coordinates (6.2) is equivalent to

$$\begin{aligned} \frac{dx_1}{dt} &= -\varepsilon(1 - z_1^2)z_2\sin t, \\ \frac{dx_2}{dt} &= \varepsilon(1 - z_1^2)z_2\cos t, \end{aligned} \tag{6.4}$$

where z_1 and z_2 are given by (6.3). The averaged system of (6.4) is then

$$\begin{aligned} \frac{dx_1}{dt} &= \frac{\varepsilon x_1}{8}[2(2 - g(A, B)) - (x_1^2 + x_2^2)], \\ \frac{dx_2}{dt} &= \frac{\varepsilon x_2}{8}[2(2 - g(A, B)) - (x_1^2 + x_2^2)], \end{aligned} \tag{6.5}$$

where $g(A, B) = A^2 + B^2/4$.

Consider two cases: (a) $g(A, B) > 2$, (b) $g(A, B) < 2$. *Case* (a). $g(A, B) > 2$. Let $g_0 = g(A, B)$. Then (6.4) is of the form $\dot{x} = \varepsilon F(t, x)$, where F is almost periodic in t with basic frequencies $\sqrt{2}$, $\sqrt{3}$. The averaged system (6.5) is of the form $\dot{x} = \varepsilon F_0(x)$. From (6.5) it is clear that the origin $x_1 = x_2 = 0$ is the only equilibrium point for (6.5), and by a simple computation the Jacobian $\partial F_0(0)/\partial x < 0$. One can now use Theorem 4.2 by taking the Liapunov function $V(x_1, x_2) = x_1^2 + x_2^2$. It is clear that $\dot{V}_{(6.5)}(x_1, x_2) = (\varepsilon/4)(x_1^2 + x_2^2)[2(2 - g_0) - (x_1^2 + x_2^2)]$. Since $2 - g_0 < 0$, $\dot{V}_{(6.5)}(x_1, x_2) < 0$ for $(x_1, x_2) \not\equiv (0, 0)$. Take $D = \{(x_1, x_2) | x_1^2 + x_2^2 < \rho_0^2\}$ for a fixed but large $\rho_0 > 0$. If $l > \rho_0^2$, then $\Omega_l = 0 = D$. By Theorem 4.2, $O \subset \Delta_0$, the domain of stability of the integral manifold for (6.4) which in this case is an almost periodic solution of (6.4) with basic frequencies $\sqrt{2}$, $\sqrt{3}$, since the θ equation is missing. In this case the results are equivalent to those obtainable by the methods of Sethna and Moran [13].

Case (b). $g(A, B) < 2$. The averaged system (6.5) now has a family of non-isolated equilibrium points given by $x_1^2 + x_2^2 = 2(2 - g_0)$. Furthermore $(0, 0)$ is also an equilibrium point of (6.5) but not of (6.4). To attack this problem return to system (6.2) and introduce the change of variables

$$\begin{aligned} z_1 &= x \sin\theta - A \sin\sqrt{2}\,t - (B/2)\sin\sqrt{3}\,t, \\ z_2 &= x\cos\theta - A\sqrt{2}\cos\sqrt{2}\,t - (B\sqrt{3}/2)\cos\sqrt{3}\,t, \end{aligned} \tag{6.6}$$

and get

$$\begin{aligned} \frac{d\theta}{dt} &= 1 - \frac{\varepsilon}{x}(1 - z_1^2)z_2\sin\theta, \\ \frac{dx}{dt} &= \varepsilon(1 - z_1^2)z_2\cos\theta, \end{aligned} \tag{6.7}$$

where z_1 and z_2 are given by (6.6). The averaged system then becomes

$$\frac{d\theta}{dt} = 1, \qquad \frac{dx}{dt} = \frac{\varepsilon x}{8}[2(2 - g_0) - x^2]. \tag{6.8}$$

The second equation in (6.8) has two equilibrium points $x = 0$ and $x = \sqrt{2(2 - g_0)}$. The Jacobian of the second equation in (6.8) is positive for $x = 0$ and negative for $x = \sqrt{2(2 - g_0)}$.

In (6.7) there is a singularity at $x = 0$ in the function associated with Θ of (2.7). Isolate x from 0 by taking $D = \{x \mid \rho_1 < x < \rho_0\}$ for $\rho_1 > 0$ and $\rho_0 \gg \sqrt{2(2 - g_0)}$. Then for ε sufficiently small, by Theorem 2.2 there exists a function $f(t, \theta, \varepsilon)$ with $f(t, \theta, 0) = \sqrt{2(2 - g_0)}$, periodic in θ of period 2π, almost periodic in t with basic frequencies $\sqrt{2}$, $\sqrt{3}$ and such that $x = f(t, \theta, \varepsilon)$ is an integral manifold of (6.7). According to Hale [5, pp. 163–164] there is also an almost periodic solution of (6.7) near 0 but for ε small this solution would fall out of $E \times D$.

Let $r = x - \sqrt{2(2 - g_0)}$. Then the second equation of (6.8) becomes

$$\frac{dr}{dt} = \frac{\varepsilon(2 - g_0)}{2}r - \frac{\varepsilon r}{8}(3\sqrt{2(2 - g_0)}\,r + r^2). \tag{6.9}$$

Let $V(r) = r^2$ and note that since $x > \rho_1$, $r > \rho_1 - \sqrt{2(2 - g_0)}$. Set

$$\Omega_l = \{r \mid v(r) < l, r > \rho_1 - \sqrt{2(2 - g_0)}\}.$$

Now

$$\dot V_{(6.9)}(r) = -\varepsilon(2 - g_0)r^2 - (\varepsilon/4)r^2(3\sqrt{2(2 - g_0)})r + r^2.$$

For $r > \rho_1 - \sqrt{2(2 - g_0)}$, $\dot V_{(6.9)}(r) < 0$. Take $l > \rho_1$. Then $O(l) = D$ and, using Corollary 4.3, $E \times D \subset \Delta_0$, the domain of stability of the integral manifold

$x = f(t, \theta, \varepsilon)$.

7. Coupled van der Pol system. Consider the system

$$\begin{aligned} \ddot{z}_1 + \mu_1^2 z_1 &= \varepsilon(1 - z_1^2 - a z_2^2)\dot{z}, \\ \ddot{z}_2 + \mu_2^2 z_2 &= \varepsilon(1 - \alpha z_1^2 - z_2^2)\dot{z}_2, \end{aligned} \tag{7.1}$$

where $\varepsilon > 0$, $a > 0$, $\alpha > 0$, and $\mu_1 > 0$, $\mu_2 > 0$ such that the nonresonance conditions $\mu_j \not\equiv 0 \pmod{\mu_k}$, $j \neq k$, and $k\mu_2 + l\mu_1 \neq 0$ for all integers k, l such that $|k| + |l| \leqq 3$, are satisfied. This system has also been studied by Hale [5, p. 165].

Transform (7.1) first by the variables $u_1 = z_1$, $u_2 = \dot{z}_1$, $w_1 = z_2$, $w_2 = \dot{z}_2$. System (7.1) becomes

$$\begin{aligned} \dot{u}_1 &= u_2, & \dot{u}_2 &= -\mu_1^2 u_1 + \varepsilon(1 - u_1^2 - a w_1^2)u_2, \\ \dot{w}_1 &= w_2, & \dot{w}_2 &= -\mu_2^2 w_1 + \varepsilon(1 - \alpha u_1^2 - w_1^2)w_2. \end{aligned} \tag{7.2}$$

Next transform (7.2) by introducing

$$\begin{aligned} u_1 &= \sqrt{x_1}\sin \mu_1\theta_1, & u_2 &= \mu_1\sqrt{x_1}\cos \mu_1\theta_1, \\ w_1 &= \sqrt{x_2}\sin \mu_2\theta_2, & w_2 &= \mu_2\sqrt{x_2}\cos \mu_2\theta_2 \end{aligned} \tag{7.3}$$

where it will be assumed that $x_1, x_2 \geqq 0$. Then (7.2) becomes

$$\begin{aligned} \dot{\theta}_1 &= 1 - (\varepsilon/2\mu_1)(\sin 2\mu_1\theta_1 - 2x_1 \sin^3 \mu_1\theta_1 \cos \mu_1\theta_1 \\ &\qquad\qquad - a x_2 \sin 2\mu_1\theta_1 \sin^2 \mu_2\theta_2), \\ \dot{\theta}_2 &= 1 - (\varepsilon/2\mu_2)(\sin 2\mu_2\theta_2 - \alpha x_1 \sin^2 \mu_1\theta_1 \sin 2\mu_2\theta_2 \\ &\qquad - 2x_2 \sin^3 \mu_2\theta_2 \cos \mu_2\theta_2), \\ \dot{x}_1 &= 2\varepsilon x_1(\cos^2 \mu_1\theta_1 - (x_1/4)\sin^2 2\mu_1\theta_1 - a x_2 \cos^2 \mu_1\theta_1 \sin^2 \mu_2\theta_2), \\ \dot{x}_2 &= 2\varepsilon x_2(\cos^2 \mu_2\theta_2 - \alpha x_1 \sin^2 \mu_1\theta_1 \cos^2 \mu_2\theta_2 - (x_2/4)\sin^2 2\mu_2\theta_2). \end{aligned} \tag{7.4}$$

This system is of the form (2.7) without the y equation. The averaged system is then given by

$$\dot{\theta}_1 = 1, \qquad \dot{\theta}_2 = 1, \tag{7.5a}$$

$$\begin{aligned} \dot{x}_1 &= \varepsilon x_1\left(1 - \frac{x_1}{4} - \frac{a x_2}{2}\right), \\ \dot{x}_2 &= \varepsilon x_2\left(1 - \frac{\alpha x_1}{2} - \frac{x_2}{4}\right). \end{aligned} \tag{7.5b}$$

The last two equations (7.5b) have been decoupled from the first two, (7.5a), which allows one to investigate the set of equilibrium points of (7.5b). These are given by

1. $x_1 = x_2 = 0$,
2. $x_1 = 0,\ x_2 = 4$,
3. $x_1 = 4,\ x_2 = 0$,
4. $x_1 = \dfrac{4 - 8a}{1 - 4a\alpha},\ x_2 = \dfrac{4 - 8\alpha}{1 - 4a\alpha}$.

Equilibrium point 4 will be the only one of interest in this example. If one translates the axes for system (7.5b) to point 4 by introducing

$$w_1 = x_1 - \left(\frac{4 - 8a}{1 - 4a\alpha}\right), \qquad w_2 = x_2 - \left(\frac{4 - 8\alpha}{1 - 4a\alpha}\right) \tag{7.6}$$

into (7.5b) one gets

$$\begin{aligned} \dot{w}_1 &= \left(\frac{2a - 1}{1 - 4a\alpha}\right) w_1 + \left(\frac{4a^2 - 2a}{1 - 4a\alpha}\right) w_2 - \frac{w_1^2}{4} - \frac{a}{2} w_1 w_2, \\ \dot{w}_2 &= \left(\frac{4\alpha^2 - 2\alpha}{1 - 4a\alpha}\right) w_1 + \left(\frac{2\alpha - 1}{1 - 4a\alpha}\right) w_2 - \frac{w_2^2}{4} - \frac{\alpha}{2} w_1 w_2. \end{aligned} \tag{7.7}$$

In order to simplify the notation somewhat let

$$\begin{aligned} p_1 &= \frac{2a - 1}{1 - 4a\alpha}, & p_2 &= \frac{4a^2 - 2a}{1 - 4a\alpha}, \\ q_1 &= \frac{4\alpha^2 - 2\alpha}{1 - 4a\alpha}, & q_2 &= \frac{2\alpha - 1}{1 - 4a\alpha}. \end{aligned} \tag{7.8}$$

A necessary and sufficient condition for the origin to be asymptotically stable for

$$\frac{dw_1}{dt} = p_1 w_1 + p_2 w_2, \qquad \frac{dw_2}{dt} = q_1 w_1 + q_2 w_2 \tag{7.9}$$

is for $\Delta = p_1 q_2 - q_1 p_2 > 0$ and $p_1 + q_2 < 0$. From (7.8),

$$\Delta = \frac{(1 - 2a)(1 - 2\alpha)}{1 - 4a}, \qquad p_1 + q_2 = \frac{2(a + \alpha - 1)}{1 - 4a\alpha}. \tag{7.10}$$

The asymptotic stability of the origin can now be characterized in terms of a and α. In particular, with a little algebra, the origin is asymptotically stable provided

1. $a > \frac{1}{2},\ 1/4a < \alpha < \frac{1}{2}$,
2. $1/4\alpha < a < \frac{1}{2},\ \frac{1}{2} < \alpha$,
3. $0 < a < \frac{1}{2},\ 0 < \alpha < \frac{1}{2}$.

Reconsidering (7.7) it is easy to see that this system can be put in the proper form for generating a Liapunov function as described in § 4. In fact, set

$$\phi_1(w_1, w_2) = h_{11}(w_1, w_2)w_1 + h_{12}(w_1, w_2)w_2,$$
$$\phi_2(w_1, w_2) = h_{21}(w_1, w_2)w_1 + h_{22}(w_1, w_2)w_2,$$

where

$$\text{(7.11)} \qquad h_{11} = -\frac{w_1}{4}, \qquad h_{12} = -\frac{a}{2}w_1$$
$$h_{21} = -\frac{\alpha}{2}w_2, \qquad h_{22} = -\frac{w_2}{4}$$

and $p_{11} = p_1, p_{12} = p_2, p_{21} = q_1, p_{22} = q_2$. Then, consider the negative definite quadratic form

$$W(w_1, w_2) = -w_1^2 - w_2^2,$$

where $c_{11} = 1$, $c_{12} = 0$, $c_{21} = 0$, $c_{22} = 1$, and solve $P^TA + AP = -C$ for the A matrix. In this case one need only solve

$$\text{(7.12)} \qquad \begin{aligned} 2a_{11}p_{11} + 2a_{12}p_{21} &= -1, \\ a_{11}p_{12} + a_{12}(p_{11} + p_{22}) + a_{22}p_{21} &= 0, \\ 2a_{12}p_{12} + 2a_{22}p_{22} &= -1, \end{aligned}$$

due to symmetry. This is a tridiagonal system and a simple though cumbersome calculation (see Arden and Astill [1, pp. 144–145]) for this tridiagonal system yields

$$\text{(7.13)} \qquad \begin{aligned} a_{11} &= -\frac{(1 + 2p_{21}a_{12})}{2p_{11}}, \\ a_{12} &= \frac{p_{12} - 2p_{11}p_{21}a_{22}}{2(p_{11}^2 + p_{11}p_{22} - p_{12}p_{21})}, \\ a_{22} &= \frac{-(p_{11}^2 + p_{11}p_{22} - p_{12}p_{21}) - p_{12}^2}{2p_{22}(p_{11}^2 + p_{11}p_{22} - p_{12}p_{21}) - 2p_{11}p_{12}p_{21}}. \end{aligned}$$

The final result is obtained by back substitution.

Note that the coefficients of the resulting Liapunov function are functions of the parameters a and α and could be obtained, if desired, by using (7.8). For illustration, let $a = \frac{1}{4}$, $\alpha = \frac{1}{4}$. Then $p_{11} = -\frac{2}{3}$, $p_{12} = -\frac{1}{3}$, $p_{21} = -\frac{1}{3}$, $p_{22} = -\frac{2}{3}$, and $a_{11} = 1$, $a_{12} = -\frac{1}{2}$, $a_{21} = -\frac{1}{2}$, $a_{22} = 1$. Then the associated V function would be

$$V(w_1, w_2) = w_1^2 - w_1w_2 + w_2^2$$

and

$$\text{(7.14)} \qquad \dot{V}_{(7.7)}(w_1, w_2) = -w^TE(w)w,$$

where w is the column vector with components w_1, w_2 and

$$e_{11} = 1 + \frac{w_1}{2} - \frac{w_2}{8},$$

$$e_{12} = e_{21} = 0,$$

$$e_{22} = 1 - \frac{w_1}{8} + \frac{w_2}{2}.$$

The criteria for negative definiteness of (7.14) are $e_{11} > \gamma$ and $e_{11}e_{22} - e_{12}e_{21} > \gamma$ for some positive γ.

Finally one can take in Corollary 4.3,

$$O(l) = \left\{ (w_1, w_2) | 1 + \frac{w_1}{2} - \frac{w_2}{8} > \gamma, \left(1 + \frac{w_1}{2} - \frac{w_2}{8}\right)\left(1 - \frac{w_1}{8} + \frac{w_2}{2}\right) > \gamma, \text{ and } w_1^2 - w_1 w_2 + w_2^2 < l \right\}.$$

In this case $E^2 \times O(l)$ is contained in the domain of stability of an integral manifold that is geometrically represented by a torus. In particular there exist functions $f_1(\theta_1, \theta_2, \varepsilon), f_2(\theta_1, \theta_2, \varepsilon)$ that are periodic in θ_1, θ_2 of periods $2\pi/\mu_1$, $2\pi/\mu_2$ such that $f_1(\theta_1, \theta_2, 0) = (4 - 8a)/(1 - 4a\alpha), f_2(\theta_1, \theta_2, 0) = (4 - 8\alpha)/(1 - 4a\alpha)$ and the integral manifold is given by

$$z_1 = \sqrt{f_1(\theta_1, \theta_2, \varepsilon)} \sin \mu_1 \theta_1, \qquad z_2 = \sqrt{f_2(\theta_1, \theta_2, \varepsilon)} \sin \mu_2 \theta_2.$$

8. Appendix. Theorem 3.5 and Lemmas 3.2, 3.3, 3.4 yield an existence of an ε^* satisfying the required conditions, but implicitly include in the proof a means for constructing this ε^*. In order to generate an algorithm for computing ε^* several quantities must be determined.

First, one must specify the set D. Then one must: (i) compute the bound M on the functions Θ, X, Y on D. (ii) Compute the Lipschitz constant λ and construct the function $h(\varepsilon)$ for uniform continuity in ε. (iii) Determine the constants c and b such that for $t \geqq 0$,

$$\|e^{AT}\| \leqq c\, e^{-bt}. \tag{8.1}$$

(iv) Pick L and μ.

Once these values have been determined then inequality (3.22) gives an estimate for v. From this estimate an appropriate value for a can be made so that (note (3.15))

$$a \leqq v/(3\lambda).$$

Once a has been fixed the function $G(\varepsilon)$ can be constructed from (3.19), given the dimensions of θ, x, y.

After determining this value one can use inequalities (3.35) and (3.38) to estimate $|\bar{\theta} - \theta|$, $|\bar{x} - x|$, $|\bar{y} - y|$ and determine ε so that

$$|\bar{\theta} - \theta| < \frac{\mu}{2}, \quad |\bar{x} - x| < \frac{\mu}{2}, \quad |\bar{y} - y| < \frac{\mu}{2}. \tag{8.2}$$

Note that

$$|\phi - \theta| \leqq |\phi - \bar{\theta}| + |\bar{\theta} - \theta| < G(\varepsilon) + \frac{\mu}{2}$$

so that first of all one needs to choose ε^* so that $0 < \varepsilon \leqq \varepsilon^*$, then

$$G(\varepsilon) \leqq \frac{\mu}{2}. \tag{8.3}$$

A similar estimate holds for $|\eta - x|$ and $|\xi - y|$.

In order to satisfy (8.2) it is sufficient that

$$\left[\frac{2cM}{b}\right]\varepsilon < \frac{\mu}{4}, \quad \left[\frac{2\lambda cL}{b}\right]\varepsilon^2 G(\varepsilon) < \frac{\mu}{4},$$

$$\left[\frac{2e^{2\lambda L}cM}{b}\right]\varepsilon < \frac{\mu}{8}, \quad \left[\frac{e^{2\lambda L}}{\lambda}\right]v < \frac{\mu}{8},$$

$$\left[\frac{e^{2\lambda L}}{\lambda}\right]h(\varepsilon) < \frac{\mu}{8}, \tag{8.4}$$

and

$$e^{2\lambda L}\left[\frac{2cL}{b} + \frac{1}{\lambda}(2\lambda + 3M) + \frac{|d(\varepsilon) - 1|}{\lambda\varepsilon} + \frac{2\|A\|k_0}{b} + \frac{\|A\|cM}{b\lambda}\right]G(\varepsilon) < \frac{\mu}{8}.$$

The inequality with v is just (3.22). This value is picked first, then ε^* is chosen.

One can now use these inequalities in an actual estimate for the van der Pol equation (5.1) for example. From § 5, if one takes the set D so that

$$D = \{\rho | 0 \leqq \rho \leqq B, 2 \ll B\},$$

then the following estimates can be obtained on $E \times D$:

$$|\Theta(\theta, \rho)| \leqq 1 + B^2, \qquad |R(\theta, \rho)| \leqq B + B^3. \tag{8.5}$$

If one sets $M = B + B^3$, then from (8.5), $|\Theta(\theta, \rho)| \leqq M$ and $|R(\theta, \rho)| \leqq M$ on $E \times D$. An estimate for the Lipschitz constant yields, for $(\theta, \rho), (\theta', \rho') \in E \times D$,

$$\begin{aligned} |\theta(\theta, \rho) - \theta(\theta', \rho')| &\leqq 4(B + 1)^2[|\theta - \theta'| + |\rho - \rho'|], \\ |R(\theta, \rho) - R(\theta', \rho'| &\leqq (2B + 4B^2)[|\theta - \theta'| + |\rho - \rho'|]. \end{aligned} \tag{8.6}$$

If one sets $\lambda = 4(B + 1)^2$, then this λ is a Lipshitz constant for both Θ and R.

If one defines the function

$$f_1(t) = \begin{cases} 2, & 0 \leqq t \leqq 1, \\ 2/t, & 1 < t, \end{cases} \tag{8.7}$$

then it is possible to show that, if $8 \leqq B$, which one can suppose since $2 \ll B$,

$$f(t) = \left(\frac{B}{4} + \frac{B^3}{16}\right) f_1(t)$$

satisfies the inequalities

$$\left|\frac{1}{t}\int_0^t \Theta^*(\theta + s, \rho)\, ds\right| \leqq f(t), \tag{8.8}$$
$$\left|\frac{1}{t}\int_0^t R^*(\theta + s, \rho)\, ds\right| \leqq f(t).$$

Then if $L > 0$ is fixed, the function

$$F(\varepsilon) = \sup_{0 \leqq s \leqq L} sf(s/\varepsilon)$$

can be computed and satisfies the estimate

$$F(\varepsilon) \leqq \left(\frac{4B + B^3}{8}\right)\varepsilon. \tag{8.9}$$

Notice that this estimate is independent of L. From (8.9) the function $G(\varepsilon)$ can be estimated as

$$G(\varepsilon) = (1 + I_a^1)F(\varepsilon),$$

where $I_a^1 \leqq 45/(16a)$, $a < \nu/(3\lambda)$ and ν is chosen, using (8.4), so that

$$\nu < [\lambda\mu/8]\varepsilon^{-2\lambda L}. \tag{8.10}$$

Since the y equation is not present in the transformed van der Pol system (5.4), the values for c and b can be ignored and only those inequalities in (8.4) that do not involve c and b need be estimated. Furthermore $h(\varepsilon) \equiv 0$ and $d(\varepsilon) \equiv 1$. Thus one needs only choose ε^* so that for $0 < \varepsilon \leqq \varepsilon^*$

$$\varepsilon^{2\lambda L}\left[\frac{2\lambda + 3M}{\lambda}\right]G(\varepsilon) < \frac{\mu}{8}. \tag{8.11}$$

In particular, if $B = 10$, say, then $M = 1010$, $\lambda = 484$. Let $L = \frac{1}{484}$ and $\mu = 10^{-3}$. Then take

$$\nu = \frac{484}{(16)(10^3)}e^{-2} \approx 4{\cdot}09 \times 10^{-3},$$

$$a = \frac{\nu}{(6)(484)} \approx 1{\cdot}41 \times 10^{-6}.$$

Then,

$$I_a^1 \leqq \frac{45}{16a} \approx 2 \times 10^6 \quad \text{and} \quad G(\varepsilon) \leqq (2 \times 10^6)F(\varepsilon)$$

or $G(\varepsilon) \leqq (2 \times 10^6)(1040)/8\ \varepsilon$. Using (8.3) and (8.11) one must choose ε so that

$$e^2\left[\frac{2(484) + 3030}{484}\right]\frac{(2 \times 10^6)(1040)}{8}\varepsilon \leqq \frac{10^{-3}}{8}$$

and

$$\frac{(2 \times 10^6)(1040)}{8}\varepsilon \leqq \frac{10^{-3}}{2}.$$

Both of these inequalities are satisfied provided

$$0 < \varepsilon \leqq \varepsilon^* = 7{\cdot}88 \times 10^{-15}.$$

These estimates point out rather graphically that the results obtained are for truly "small" parameter problems. It would be an interesting problem to compute hard estimates for all of the inequalities in the proofs to determine the extent to which they could be used for numerical approximations. Even if the computational efficiency might be determined to be questionable this would be a worthwhile fact to know.

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REFERENCES

[1] B. W. ARDEN AND K. N. ASTILL, *Numerical Algorithms: Origins and Applications*, Addison-Wesley, Reading, Mass., 1970.

[2] N. N. BOGOLIUBOV AND Y. A. MITROPOLSKY, *Asymptotic Methods in the Theory of Non-linear Oscillations*, Gordon and Breach, New York, 1961.

[3] H. BOHR, *Almost Periodic Functions*, Chelsea, New York, 1951.

[4] D. E. GILSINN, *Domains of stability for integral manifolds*, Differential Equations and Dynamical Systems, II, J. A. Yorke, ed., Springer-Verlag, Berlin, 1970.

[5] J. K. HALE, *Oscillations in Nonlinear Systems*, McGraw-Hill, New York, 1963.

[6] ———, *Integral manifolds of perturbed differential systems*, Ann. of Math., 73 (1961), pp. 496–531.

[7] J. K. HALE AND A. P. STOKES, *Behavior of solutions near integral manifolds*, Arch. Rational Mech. Anal., 6 (1960), pp. 130–167.

[8] N. N. KRASOVSKII, *Stability of Motion*, Stanford University Press, Stanford, Calif., 1963.

[9] N. KRYLOV AND N. BOGOLIUBOV, *Introduction to Non-linear Mechanics*, Princeton University Press, Princeton, N.J., 1947.

[10] J. LASALLE AND S. LEFSCHETZ, *Stability by Liapunov's Direct Method*, Academic Press, New York, 1961.

[11] W. S. LOUD AND P. R. SETHNA, *Some explicit estimates for domains of attraction*, J. Differential Equations, 2 (1966), pp. 158–172.

[12] P. R. SETHNA, *Domains of attraction of some quasiperiodic solutions*, Differential Equations and Dynamic Systems, Academic Press, New York, 1967, pp. 323–332.

[13] P. R. SETHNA AND T. J. MORAN, *Some nonlocal results for weakly nonlinear dynamical systems*, Quart. Appl. Math., 26 (1968), pp. 175–185.

[14] E. T. WHITTAKER AND G. N. WATSON, *A Course of Modern Analysis*, Cambridge University Press, London, 1962.

[15] V. I. ZUBOV, *Methods of A. M. Liapunov and Their Applications*, P. Noordhoff, Groningen, the Netherlands, 1964.